Kripke completeness, canonicity, and quasi-canonicity in first-order modal logic

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What this talk is about

This is an introductory talk aimed at the audience with a very basic familiarity with first-order modal logic (FOML).

We'll be talking about *completeness proofs wrt to Kripke semantics*.

We'll discuss whether *canonicity*, as conceived in propositional modal logic, is the right concept for FOMLs.

We'll also discuss the recently introduced concept of *quasi-canonicity*.

Two views of FOML

It appears that there are broadly two views of FOML in the literature:

1. Semantically motivated: start with (some kind of) Kripke semantics (= 'propositional' Kripke frames with domains) for a first-order modal language and define logics as sets of formulas valid on those frames.
2. Syntactically motivated: define FOMLs as minimal logics (= sets of formulas closed under certain inference rules) extending both the classical first-order logic **QCL** and propositional modal logics; e.g., define first-order **S4** to be the smallest logic containing both **QCL** and propositional **S4**.

Two views of FOML (contd.)

The first approach often leads to logics that are not conservative extensions of **QCL** since some natural classes of Kripke frames with domains do not validate classical validities with free variables; see, e.g.,

- M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. 2nd ed. Springer, 2023.

Such logics are often called *Kripkean*.

We shall here call logics arising under the second approach *classical* since they are (conservative) extensions of **QCL**. (Note that the term 'classical modal logic' has a different meaning in the literature.) This approach leads to the question of the adequate semantics for classical FOMLs; see, e.g.,

- D. Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Non-classical Logics*. Elsevier, 2009.

In particular, there is the question of which logics can be adequately characterized within Kripke semantics.

Two views of FOML (contd.)

This talk is devoted to *classical* FOMLs.

We'll be talking about completeness proofs with respect to Kripke semantics (i.e., Kripke frames with domains) for these logics.

Kripke semantics is the simplest semantics for classical FOMLs (just as it is for Kripkean logics).

Syntax

First-order modal formulas ($\mathcal{ML}$-formulas):

$$\varphi \ := \ P(x_1, \ldots, x_n) \mid \bot \mid (\varphi \to \varphi) \mid \forall x\, \varphi \mid \Box\varphi,$$

where P is an n-ary predicate letter. Nullary letters (i.e, $n = 0$) are *proposition letters*; we write them without parentheses: P, Q, etc.

Standard abbreviations:

$$\begin{array}{lcl}
\neg\varphi & := & \varphi \to \bot; \\
\varphi \vee \varphi & := & \neg\varphi \to \psi; \\
\varphi \wedge \varphi & := & \neg(\varphi \to \neg\psi); \\
\varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi); \\
\exists x\, \varphi & := & \neg\forall x\, \neg\varphi; \\
\Diamond\varphi & := & \neg\Box\neg\varphi.
\end{array}$$

Hilbert systems

Minimal system **QK**:

- Axioms of **QCL**.
- A propositional modal axiom:
  (K) $\Box(P \to Q) \to (\Box P \to \Box Q)$.
- Modus ponens (MP):
  $$\frac{\varphi, \varphi \to \psi}{\psi}.$$
- Predicate Substitution (Sub):
  $$\frac{\varphi}{\varphi'} \quad \text{if } \varphi' \text{ is a substitution instance of } \varphi.$$
- Generalisation (Gen):
  $$\frac{\varphi}{\forall x\, \varphi}.$$
- Necessitation (N):
  $$\frac{\varphi}{\Box \varphi}.$$

Other systems

Other systems are obtained by adding axioms to **QK**.

- A common case: add to **QK** propositional formulas axiomatizing a propositional modal logic Λ (call resultant system $\mathbf{Q}\Lambda$); e.g.,

$$
\begin{array}{llll}
\mathbf{QT} & := & \mathbf{QK} & \oplus \;\; \Box P \to P; \\
\mathbf{QD} & := & \mathbf{QK} & \oplus \;\; \Diamond \top; \\
\mathbf{QKB} & := & \mathbf{QK} & \oplus \;\; P \to \Box\Diamond P; \\
\mathbf{QK4} & := & \mathbf{QK} & \oplus \;\; \Box P \to \Box\Box P; \\
\mathbf{QS4} & := & \mathbf{QT} & \oplus \;\; \mathbf{QK4}; \\
\mathbf{QK5} & := & \mathbf{QK} & \oplus \;\; \Diamond\Box P \to \Box P; \\
\mathbf{QS5} & := & \mathbf{QS4} & \oplus \;\; \Diamond\Box P \to \Box P; \\
\mathbf{QGL} & := & \mathbf{QK4} & \oplus \;\; \Box(\Box P \to P) \to \Box P.
\end{array}
$$

- Axioms containing both quantifiers and modalities (here, $bf = \forall x\, \Box P(x) \to \Box \forall x\, P(x)$, the Barcan formula):

$$
\mathbf{QK} \oplus bf;
$$
$$
\mathbf{QK} \oplus \Box bf;
$$
$$
\ldots
$$

It is convenient to abstract away from Hilbert-style calculi and introduce an abstract notion of a first-order modal logic:

Definition

A (classical normal) first-order modal logic (henceforth, logic) is a set of $\mathcal{ML}$-formulas that

- includes **QCL**;
- includes the minimal normal propositional modal logic **K**;
- is closed under (MP), (Sub), (Gen), and (Nec).

The minimal logic is called **QK**.

Definition

- If L is a logic and Γ a set of $\mathcal{ML}$-formulas, then $L \oplus \Gamma$ is the smallest logic including $L \cup \Gamma$.
- If Λ is a propositional modal logic, then $\mathbf{Q}\Lambda := \mathbf{QK} \oplus \Lambda$ (*the first-order counterpart of* Λ).

Kripke semantics

A Kripke frame is a pair $\mathfrak{F} = \langle W, R \rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$.

A Kripke frame with expanding domains is a tuple $\boldsymbol{F} = \langle W, R, D \rangle$ where $\langle W, R \rangle$ is a Kripke frame and D is a system $(D_w)_{w \in W}$ of domains, i.e., a family of non-empty sets satisfying the following condition:

$$wRw' \implies D_w \subseteq D_{w'}. \qquad (ED)$$

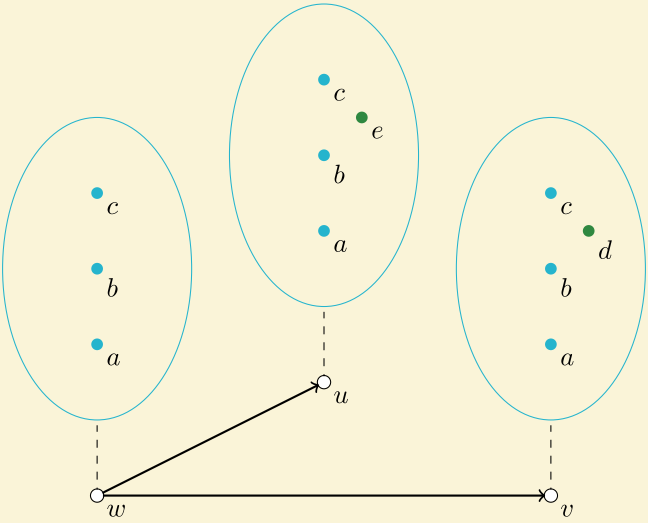
A Kripke model is a tuple $\mathfrak{M} = \langle W, R, D, V \rangle$, where V is a family $(V_w)_{w \in W}$ of valuations, where each valuation V_w is a classical valuation on D_w (in other words, for every $w \in W$, the pair $\langle D_w, V_w \rangle$ is a classical model, i.e., $V_w(P) \subseteq D_w^n$, for every n-ary P).

A Kripke frame with (locally) constant domains is a Kripke frame with expanding domains satisfying a condition that is stronger than (ED):

$$wRw' \implies D_w = D_{w'}. \qquad (CD)$$

NB The latter semantics is equivalent to the semantics of Kripke frames with a single domain shared by all worlds.

Kripke frame with expanding domains



Example

$$
\begin{aligned}
D_0 &= \{(j)ohn, (m)ary\}, \\
V_0(Loves) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\
V_0(Married) &= \varnothing, \\
\mathfrak{M}_0 &= \langle D_0, V_0\rangle.
\end{aligned}
$$

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\end{aligned}
$$

A year later...

$$
\begin{aligned}
D_1 &= \{(j)ohn, (m)ary\}, \\
V_1(Loves) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\
V_1(Married) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\
\mathfrak{M}_1 &= \langle D_1, V_1\rangle.
\end{aligned}
$$

Example

$$\begin{aligned} D_0 &= \{(j)ohn, (m)ary\}, \\ V_0(Loves) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\ V_0(Married) &= \varnothing, \\ \mathfrak{M}_0 &= \langle D_0, V_0\rangle. \end{aligned}$$

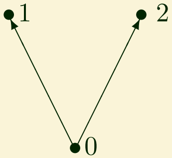
A year later...

$$\begin{aligned} D_1 &= \{(j)ohn, (m)ary\}, \\ V_1(Loves) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\ V_1(Married) &= \{\langle j, m\rangle, \langle m, j\rangle\}, \\ \mathfrak{M}_1 &= \langle D_1, V_1\rangle. \end{aligned}$$

Or maybe ...

$$\begin{aligned} D_2 &= \{(j)ohn, (m)ary, (s)teeve\}, \\ V_2(Loves) &= \{\langle j, m\rangle, \langle m, s\rangle, \langle s, m\rangle\}, \\ V_2(Married) &= \{\langle s, m\rangle, \langle m, s\rangle\}, \\ \mathfrak{M}_2 &= \langle D_2, V_2\rangle. \end{aligned}$$

Example



$W = \{0, 1, 2\},\ R = \{\langle 0, 1\rangle, \langle 0, 2\rangle\};$

$\mathfrak{F} = \langle W, R\rangle;$

$\mathfrak{M} = \langle \mathfrak{F}, D, V\rangle.$

$D_1 = \{j, m\},\ \ D_2 = \{j, m\},\ \ D_3 = \{j, m, s\},$

$$\begin{array}{lcl}
V_0(\mathit{Loves}) & = & \{\langle j, m\rangle, \langle m, j\rangle\}; \\
V_0(\mathit{Married}) & = & \varnothing; \\
V_1(\mathit{Loves}) & = & \{\langle j, m\rangle, \langle m, j\rangle\}; \\
V_1(\mathit{Married}) & = & \{\langle j, m\rangle, \langle m, j\rangle\}; \\
V_2(\mathit{Loves}) & = & \{\langle j, m\rangle, \langle m, s\rangle, \langle s, m\rangle\}; \\
V_2(\mathit{Married}) & = & \{\langle s, m\rangle, \langle m, s\rangle\}.
\end{array}$$

Evaluating formulas:

$$\begin{array}{l}
\mathfrak{M}, 0 \models \exists x \exists y\, (\mathit{Loves}(x, y) \wedge \mathit{Loves}(y, x)); \\
\mathfrak{M}, 0 \models \exists x \exists y\, \Diamond\, (\mathit{Loves}(x, y) \wedge \mathit{Loves}(y, x)); \\
\mathfrak{M}, 0 \models \Box\, \exists x \exists y\, (\mathit{Loves}(x, y) \wedge \mathit{Loves}(y, x)); \\
\mathfrak{M}, 0 \not\models \exists x \exists y\, \Box\, (\mathit{Loves}(x, y) \wedge \mathit{Loves}(y, x)).
\end{array}$$

Kripke semantics: satisfaction

Let $\boldsymbol{F} = \langle W, R, D \rangle$ be a Kripke frame with expanding domains and $w \in W$. A D_w-sentence is an expression obtained from an $\mathcal{ML}$-formula by substituting (copies of) elements of D_w for free variables of the formula.

The satisfaction relation $\Vdash$ between models $\mathfrak{M}$, worlds w, and D_w-sentences φ is defined by recursion:

$$
\begin{array}{lcl}
\mathfrak{M}, w \Vdash P(a_1, \ldots, a_n) & \leftrightharpoons & \langle a_1, \ldots, a_n \rangle \in V_w(P); \\
\mathfrak{M}, w \nVdash \bot; & & \\
\mathfrak{M}, w \Vdash \varphi \to \psi & \leftrightharpoons & \mathfrak{M}, w \nVdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi; \\
\mathfrak{M}, w \Vdash \forall x\, \varphi & \leftrightharpoons & \mathfrak{M}, w \Vdash \varphi(a), \text{ for every } a \in D_w; \\
\mathfrak{M}, w \Vdash \Box\varphi & \leftrightharpoons & \mathfrak{M}, w' \Vdash \varphi, \text{ for every } w' \in R(w).
\end{array}
$$

Validity:

$$
\begin{array}{lcl}
\mathfrak{M} \models \varphi & \leftrightharpoons & \mathfrak{M}, w \Vdash \bar{\forall}\varphi, \text{ for every } w \in W; \\
\boldsymbol{F} \models \varphi & \leftrightharpoons & \langle \boldsymbol{F}, V \rangle \models \varphi, \text{ for every } V; \\
\mathfrak{F} \models \varphi & \leftrightharpoons & \langle \boldsymbol{F}, D \rangle \models \varphi, \text{ for every } D.
\end{array}
$$

The set of formulas valid on all Kripke frames with expanding domains coincides with **QK**.

Completeness proofs

Completeness proofs with respect to Kripke semantics combine the canonical model method for propositional modal logic with the Henkin-style completeness proof for the classical first-order logic.

As in propositional logic, the worlds of a canonical model are maximally consistent theories. What should the domains be?

One option is to use individual variables as elements of domains.

It is more general (out of cardinality considerations), and conceptually perhaps clearer, to enrich the language with constants and make them elements of domains. The additional constants are a technical tool, they will not show up in the theorems we prove.

Constants and theories

We add to $\mathcal{ML}$ a countable set $\mathcal{C}^*$ of constants. (We work with the case $|\mathcal{C}^*| = \aleph_0$ for simplicity; generalisation to the case $|\mathcal{C}^*| = \kappa$, with κ an arbitrary infinite cardinal, is straightforward.)

We denote constants by $a, b, c, \ldots$.

Definition

A theory is a set of sentences, possibly with constants from $\mathcal{C}^*$.

Definition

The set of constants occurring in a theory Γ is denoted by $\mathcal{C}_\Gamma$; the set of all sentences with constants from $\mathcal{C}_\Gamma$ is called the language of Γ and denoted by $\mathcal{L}(\Gamma)$.

Derivability from theories

Definition

Let L be a logic and Γ a theory. A formula φ is *L-derivable* from Γ (notation: $\Gamma \vdash_L \varphi$) if there exists a finite sequence of formulas whose every member is

- an element of $L \cup \Gamma$;
- obtained from preceding members of the sequence by (MP).

If we were dealing Hilbert systems rather than abstract logics, we would restrict application of (Sub) and (N) to formulas obtained using only axioms, and no elements of Γ.

Theorem (Deduction theorem)

For every logic L, theory Γ, sentence φ, and formula ψ,

$$\Gamma, \varphi \vdash_L \psi \quad \Longleftrightarrow \quad \Gamma \vdash_L \varphi \to \psi.$$

L-consistent theories

Definition

A theory Γ is L-inconsistent if $\Gamma \vdash_L \bot$. Otherwise, Γ is L-consistent.

By Deduction theorem, a theory Γ is L-inconsistent if there exist $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_L \neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$; otherwise, Γ is L-consistent.

Completeness with respect to Kripke semantics

Definition

A logic L is strongly Kripke complete if every L-consistent $\mathcal{ML}$-theory (no constants!) is satisfiable in a Kripke model over a Kripke frame with expanding domains validating L.

Definition

A logic L is Kripke complete if there exists a class $\mathscr{C}$ of Kripke frames with expanding domains such that, for every $\mathcal{ML}$-formula φ (no constants!),

$$\varphi \in L \iff \mathscr{C} \models \varphi.$$

Worlds of canonical models

In canonical models for propositional models, worlds are simply maximal consistent sets of formulas.

In FOMLs, they are theories with special properties. To define them, we need the following notions:

- L-complete theory;
- Henkin theory;
- L-place (= an L-complete Henkin theory with an additional requirement).

Definition

A theory Γ is L-complete if Γ is L-consistent, but every theory Δ such that $\Gamma \subset \Delta \subseteq \mathcal{L}(\Gamma)$ is L-inconsistent.

Lemma

For every L-consistent theory Γ, the following conditions are equivalent:

- Γ is L-complete;
- for every $\varphi \in \mathcal{L}(\Gamma)$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Lemma

If Γ is an L-complete theory and $\varphi \in \mathcal{L}(\Gamma)$, then $\Gamma \vdash_L \varphi \iff \varphi \in \Gamma$.

Lemma

If Γ is an L-complete theory, then, for every $\varphi, \psi \in \mathcal{L}(\Gamma)$,

- $\varphi \wedge \psi \in \Gamma \iff \varphi \in \Gamma$ and $\psi \in \Gamma$;
- $\varphi \vee \psi \in \Gamma \iff \varphi \in \Gamma$ or $\psi \in \Gamma$;
- $\varphi \to \psi \in \Gamma \iff \varphi \notin \Gamma$ or $\psi \in \Gamma$.

In particular, L-complete theories are closed under (MP).

Lemma (Lindenbaum lemma)

Let Γ be an L-consistent theory. Then, there exists an L-complete theory Δ such that $\Gamma \subseteq \Delta \subseteq \mathcal{L}(\Gamma)$.

Henkin theories

Definition

A theory Γ is Henkin if, for every $\exists x\, \varphi(x) \in \mathcal{L}(\Gamma)$, there exists $a \in \mathcal{C}_\Gamma$ such that $\exists x\, \varphi(x) \rightarrow \varphi(a) \in \Gamma$.

Lemma (Witness property)

If Γ is a Henkin theory and $\exists x\, \varphi(x) \in \mathcal{L}(\Gamma)$, then

$$\exists x\, \varphi(x) \in \Gamma \quad \Longleftrightarrow \quad \exists a \in \mathcal{C}_\Gamma\ \varphi(a) \in \Gamma. \qquad \text{(WP)}$$

Lemma (Henkin lemma)

Every L-consistent theory can be extended to a Henkin theory (in a language with countably many additional constants).

L-places

Definition

A set $\mathcal{C} \subseteq \mathcal{C}^*$ is small if $\mathcal{C}^* - \mathcal{C}$ is infinite.

Definition

An L-place is an L-complete Henkin theory Γ such that $\mathcal{C}_\Gamma$ is small.

Using Lindenbaum lemma, Henkin lemma, and cardinality considerations, we obtain the following:

Lemma (L-place lemma)

Every L-consistent theory with a small set of constants can be extended to an L-place.

Canonical models and frames

Definition

The canonical model for L with respect to $\mathcal{C}^*$ is the tuple $\mathfrak{M}_L = \langle W_L, R_L, D_L, V_L \rangle$ where

- W_L is the set of all L-places;
- $R_L \subseteq W_L \times W_L$ is the canonical accessibility relation:

$$\Gamma R_L \Delta \quad \leftrightharpoons \quad \Box^- \Gamma \subseteq \Delta, \quad \text{where } \Box^- \Gamma := \{\varphi \mid \Box\varphi \in \Gamma\}.$$

- $D_L \colon W_L \to \mathcal{C}^*$ is the canonical system of domains:

$$D_L(\Gamma) := \mathcal{C}_\Gamma;$$

- V_L is the canonical valuation:

$$(V_L)_w(P^n) \quad := \quad \{\boldsymbol{c} \in \mathcal{C}^n_\Gamma \mid P^n(\boldsymbol{c}) \in \Gamma\}.$$

Lemma

For every $\Gamma, \Delta \in W_L$,

$$\Gamma R_L \Delta \Longrightarrow \mathcal{C}_\Gamma \subseteq \mathcal{C}_\Delta.$$

Proof. Let $a \in \mathcal{C}_\Gamma$. Then, $\Box(P(a) \to P(a)) \in \mathcal{L}(\Gamma)$.

Since Γ is L-complete and $\Gamma \vdash_L \Box(P(a) \to P(a))$, it follows that $\Box(P(a) \to P(a)) \in \Gamma$.

Since $\Gamma R_L \Delta$, the definition of R_L implies that $(P(a) \to P(a)) \in \Delta$, and so $a \in \mathcal{C}_\Delta$.

Definition

The tuple $\boldsymbol{F}_L := \langle W_L, R_L, D_L \rangle$ is the canonical Kripke frame with expanding domains.

Lemma (Existence lemma)

Let $\boldsymbol{F}_L = \langle W_L, R_L, D_L \rangle$ be the canonical Kripke frame with expanding domains and let $\Gamma \in W_L$, with $\Diamond\varphi \in \Gamma$. Then, there exists an L-complete theory Δ such that

- $\mathcal{L}(\Gamma) = \mathcal{L}(\Delta)$;
- $\Gamma R_L \Delta$;
- $\varphi \in \Delta$.

Truth lemma and canonical model theorem

Lemma (Truth lemma)

Let $\mathfrak{M}_L = \langle W_L, R_L, D_L, V_L \rangle$ be a canonical model and $\Gamma \in W_L$. Then, for every $\varphi \in \mathcal{L}(\Gamma)$,

$$\mathfrak{M}_L, \Gamma \Vdash \varphi \iff \varphi \in \Gamma.$$

Proof. By induction on φ.

- Case $\varphi = P(a_1, \ldots, a_n)$: by definition of $\mathfrak{M}_L$.
- Boolean cases: by properties of L-complete sets.
- Case $\varphi = \exists x\, \psi$: by (WP).
- Case $\varphi = \Diamond \psi$: by Existence lemma.

Theorem (Canonical model theorem)

Let $\mathfrak{M}_L$ be a canonical model for L. Then, for every $\varphi \in \mathcal{ML}$,

$$\varphi \in L \iff \mathfrak{M}_L \models \varphi.$$

Definition

A logic L is canonical if it is valid on its canonical Kripke frame with expanding domains, i.e., if $\boldsymbol{F}_L \models L$.

Theorem

Every canonical logic is strongly Kripke complete.

Proof.
Let L be canonical and let Γ_0 be an L-consistent theory.

Then, $\Gamma_0 \subseteq \Gamma$, for some L-place Γ. By Truth lemma, $\mathfrak{M}_L, \Gamma \Vdash \Gamma_0$.
Since L is canonical, $\boldsymbol{F}_L \models L$.

Theorem

The logic **QK** is canonical and, therefore, strongly Kripke complete.

Canonical formulas

Definition

A formula φ is canonical if the logic $\mathbf{QK} \oplus \varphi$ is canonical.

Proposition

If L is a canonical logic and φ is a canonical formula, then the logic $L \oplus \varphi$ is canonical.

Examples of canonical formulas

Definition

A formula is constant if it is built from $\bot$ using $\rightarrow$ and $\Box$.

Definition

A formula is one-way pseudo-transitive if it has the form $\Box\varphi \rightarrow \Box^n\varphi$, for some $n \geqslant 0$.

Examples:

(D)	$\Diamond\top$	constant;
(T)	$\Box P \rightarrow P$	one-way pseudo-transitive;
(4)	$\Box P \rightarrow \Box\Box P$	one-way pseudo-transitive.

Examples of canonical formulas (contd.)

Theorem

Every constant formula is canonical.

Theorem

Every one-way pseudo-transitive formula is canonical.

Corollary

If $L = \mathbf{QK} \oplus \Gamma$, where Γ is a set of closed or one-way pseudo-transitive formulas, then L is canonical and, hence, strongly Kripke complete.

Corollary

Logics **QD**, **QT**, **QK4**, and **QS4** are strongly Kripke complete.

These logics are complete wrt to the expected classes of Kripke frames.

Failure of transfer of canonicity

Canonicity of a propositional logic Λ does not guarantee canonicity of $\mathbf{Q}\Lambda$.

Example 1: KB is canonical, but **QKB** is not, since the formula $B = P \to \Box\Diamond P$ is canonical propositionally, but not in FOML:

Propositional reasoning: Suppose that $\Gamma R_L \Delta$, but not $\Delta R_L \Gamma$. Then, there exists a formula φ such that $\Box\varphi \in \Delta$, but $\varphi \notin \Gamma$. Then, $\neg\varphi \in \Gamma$. Since Γ is an L-complete set, $\neg\varphi \to \Box\Diamond\neg\varphi \in \Gamma$. Hence, $\Box\Diamond\neg\varphi \in \Gamma$, and so $\Diamond\neg\varphi \in \Delta$, in contradiction with $\Box\varphi \in \Delta$.

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This reasoning fails in the FO case since $\varphi \in \mathcal{L}(\Delta)$, but not necessarily $\varphi \in \mathcal{L}(\Gamma)$ (recall that $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Delta$, but not necessarily vice versa): problematic steps are in blue.

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We can turn this observation into a proof:

Lemma

For every L and every $\Gamma, \Delta \in W_L$, if $\Gamma R_L \Delta$, then $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Delta$.

Let $L = \mathbf{QK} + B\,(= \mathbf{QKB})$. Then, $\Diamond\top$ is L-consistent (it is satisfied on a Kripke frame consisting of a single reflexive world). Hence, there exists an L-place Γ such that $\Diamond\top \in \Gamma$.

By Existence lemma, $\Box^-\Gamma \cup \{\top\}$ is L-consistent. Let $a \in \mathcal{C}^* - \mathcal{C}_\Gamma$ (such an a exists since Γ is an L-place, and so $\mathcal{C}_\Gamma$ is small) and let P be a monadic predicate letter. Since $\Box^-\Gamma \cup \{\top\}$ is L-consistent, so is $\Box^-\Gamma \cup \{P(a) \to P(a)\}$.

Hence, there exists an L-place Δ such that $\Box^-\Gamma \cup \{P(a) \to P(a)\} \subseteq \Delta$. Since $\Box^-\Gamma \subseteq \Delta$, it follows that $\Gamma R_L \Delta$.

On the other hand, $\mathcal{C}_\Delta \not\subseteq \mathcal{C}_\Gamma$; hence, by Lemma, $\Delta R_L \Gamma$ does not hold. Thus, R_L is not symmetric. Hence $\boldsymbol{F}_L \not\models \mathbf{QKB}$.

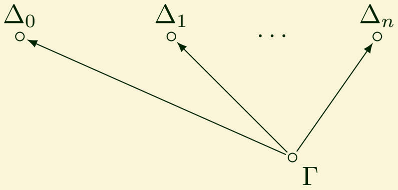
Example 2: $\mathbf{KAlt}_n$ is canonical, but $\mathbf{QAlt}_n$ is not, since the formula

$$\boldsymbol{alt}_n \quad := \quad \neg \bigwedge_{i=0}^{n} \Diamond(P_i \wedge \bigwedge_{j \neq i} \neg P_j)$$

is canonical propositionally, but not in FOML.

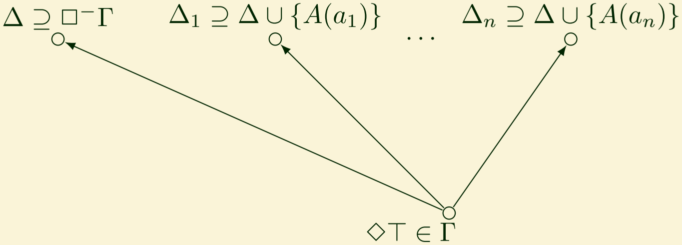
Propositional reasoning: If $i \neq j$, there exists φ_{ij} such that $\varphi_{ij} \in \Delta_i$ and $\varphi_{ij} \notin \Delta_j$. Put $\psi_i := \bigwedge_{j \neq i} \varphi_{ij}$. Then, $\psi_i \in \Delta_j$ iff $i = j$. Thus, $\psi_i \wedge \bigwedge_{j \neq i} \neg \psi_j \in \Delta_i$. Hence, $\Diamond(\psi_i \wedge \bigwedge_{j \neq i} \neg \psi_j) \in \Gamma$, for every $i \in \{1, \ldots, n\}$.

By $\boldsymbol{alt}_n$, $\neg \bigwedge_{i=0}^{n} \Diamond(\psi_i \wedge \bigwedge_{j \neq i} \neg \psi_j) \in \Gamma$. This gives us a contradiction.



Failure of transfer of canonicity (contd.)

This fails in FOML since $\psi_i \notin \mathcal{L}(\Gamma)$. This observation can be turned into a proof:



Let $L = \mathbf{QAlt}_n$. Choose $a_1, \ldots, a_n \in \mathcal{C}^* - \mathcal{C}_\Delta$.

Put $\psi(a_i) := P(a_i) \to P(a_i)$. Extend each set $\Delta \cup \{\psi(a_i)\}$ to an L-place Δ_i so that $\mathcal{C}_{\Delta_i} \subseteq \mathcal{C}^* - \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$.

Then, $\Delta_i \neq \Delta_j$ whenever $i \neq j$. Hence, $\boldsymbol{F}_L \not\models alt_n$, and so $\boldsymbol{F}_L \not\models L$.

Quasi-canonicity

It looks like, in general, FO canonical models contain too much "junk".

Recently, Valentin Shehtman introduced the notion of a quasi-canonical model and a quasi-canonical logic. Quasi-canonical models are obtained from canonical models by selection that resembles selective filtration in propositional modal logic.

Selective weak submodels

Definition

A Kripke model $\mathfrak{M}' = (W', R', D', V')$ is a weak submodel of a Kripke model $\mathfrak{M} = (W, R, D, V)$ if $W' \subseteq W$, $R' \subseteq R$, and, for every $w \in W'$, both $D_w = D'_w$ and $V'_w = V_w$. If, additionally,

$$\mathfrak{M}, w \Vdash \Diamond\varphi \quad \Longrightarrow \quad \exists v \in R'(w)\ \mathfrak{M}, v \Vdash \varphi,$$

for every $w \in W'$ and every D_w-sentence φ, then $\mathfrak{M}'$ is a selective weak submodel of $\mathfrak{M}$ (notation: $\mathfrak{M}' \Subset \mathfrak{M}$).

Note that weak submodels differ from submodels: in weak submodels it is possible that $R' \subset R \restriction W'$, while in submodels $R' = R \restriction W'$.

Lemma

Let $\mathfrak{M}' \Subset \mathfrak{M}$ and let w be a world of $\mathfrak{M}'$. Then, for every $w \in W'$ and every D_w-sentence φ,

$$\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}', w \Vdash \varphi.$$

Quasi-canonical logics

Definition

A quasi-canonical model for a logic L is a selective weak submodel of the canonical model for L.

Definition

A logic L is quasi-canonical if, for every L-place Γ, there exists a quasi-canonical model over a Kripke frame with expanding domains (W, R, D) such that $\Gamma \in W$ and $(W, R, D) \models L$.

Theorem

Every quasi-canonical logic is strongly Kripke complete.

Proof. Immediate from definitions: the existence of a weak selective submodel over the right Kripke frame with domains proves strong Kripke completeness.

Example: $\mathbf{QAlt}_n$

Theorem

Let $L = \mathbf{QAlt}_n$, for some $n \geqslant 1$. Then, L is quasi-canonical and, hence, strongly Kripke complete.

Proof. Let $\mathfrak{M}_L = (W_L, R_L, D_L, V_L)$ be a canonical model for L and let $\Gamma_0 \in W_L$. We obtain a model $\mathfrak{M} \Subset \mathfrak{M}_L$ over a Kripke frame with expanding domains validating L and containing Γ_0. The key observation is the following (the proof is similar to the propositional reasoning for $\mathbf{Alt}_n$):

Lemma

If $\Gamma \in W_L$ *and*

$$X^\Gamma = \{\Delta \mid \Delta \textit{ is an } L\textit{-complete theory } \& \ \mathcal{L}(\Delta) = \mathcal{L}(\Gamma) \ \& \ \Box^-\Gamma \subseteq \Delta\},$$

then $|X^\Gamma| \leqslant n$.

This is enough to apply Existence lemma to X^Γ rather than to W_L.

Example: $\mathbf{QAlt}_n$

Hence, we can build the selective weak submodel of $\mathfrak{M}_L$ containing Γ_0 by recursion, starting with Γ_0 and selecting its successors based on Lemma. The lemma makes sure that all the diamonds are realized, and hence 'Truth Lemma is still effective'.

Essentially the same argument works for $\mathbf{QTAlt}_n$.

This argument can be generalised to logics of finite uniform trees and to a few other situations.

Example: Logics with the Barcan formula

The logic **QKTB** proves the Barcan formula $bf = \forall x \,\Box P(x) \to \Box \forall x \, P(x)$:

Lemma

If $L \vdash P \to \Box\Diamond P$, then $L \vdash bf$.

The formula bf is valid precisely on Kripke frames with locally constant domains.

Canonical models for these logics do *not* satisfy the local constancy condition (again, they contain too much 'junk'). We can, however, select a weak selective submodel of their canonical model based on a Kripke frame with locally constant domains (in fact, with a constant domain).

We can only prove (weak) completeness this way. The key observation is the following:

Example: Logics with the Barcan formula (contd.)

Lemma (Existence lemma for constant domains)

Let L be logic containing bf. If $\Gamma \in W_L$ and $\Diamond\varphi \in \Gamma$, then there exists $\Delta \in W_L$ such that

- $\mathcal{L}(\Gamma) = \mathcal{L}(\Delta)$;
- $\Gamma R_L \Delta$;
- $\varphi \in \Delta$.

NB Notice the difference with Existence Lemma (for expanding domains): here Δ is Henkin; the proof goes through due to bf.

By Lemma above, $L \vdash p \to \Box\Diamond p$ implies $L \vdash bf$, hence we can do the following:

Lemma

Let L be a logic such that $L \vdash p \to \Box\Diamond p$ and let $\mathfrak{M} = \langle W, R, D, V \rangle$ be a quasi-canonical model for L built using Existence lemma for constant domains. Then, R is symmetric.

Proof. *Essentially, propositional reasoning*: Suppose $\Gamma R \Delta$. For contradiction, assume that $\Box\varphi \in \Delta$, but $\varphi \notin \Gamma$. Since $\Gamma, \Delta \in W$, it follows that $\mathcal{L}(\Gamma) = \mathcal{L}(\Delta)$. Since $\varphi \in \mathcal{L}(\Delta)$ and $\mathcal{L}(\Gamma) = \mathcal{L}(\Delta)$, it follows that $\varphi \in \mathcal{L}(\Gamma)$. Hence, $\neg\varphi \in \Gamma$. By (Sub), $L \vdash \neg\varphi \to \Box\Diamond\neg\varphi$, and so $\neg\varphi \to \Box\Diamond\neg\varphi \in \Gamma$. But then $\Diamond\neg\varphi \in \Delta$, i.e. $\neg\Box\varphi \in \Delta$, contrary to L-consistency of Δ.

Corollary

QKB is Kripke complete.

(Weak) completeness for logics with Barcan

Theorem (Follows from Tanaka and Ono 2001)

Let Λ be a universal Kripke complete propositional logic (= complete wrt a class of frames definable by universal classical first-order sentences) such that $\mathbf{Q}\Lambda \vdash bf$. Then, $\mathbf{Q}\Lambda$ is Kripke complete.

Corollary

QS5 is Kripke complete.

General results on quasi-canonicity

There are a handful of general results on quasi-canonicity:

Theorem

If a logic L is quasi-canonical and Γ is a set of constant formulas, then $L \oplus \Gamma$ is quasi-canonical.

Theorem

If a logic L is quasi-canonical and Γ is a pure equality theory, then $L \oplus \Gamma$ is quasi-canonical.

Theorem

If φ is a single-variable propositional formula of modal depth 1, then $\mathbf{QK} \oplus \varphi$ is quasi-canonical.

Thank you!

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