Some prospects for semiproducts and products of modal logics

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Abstract

We consider products and semiproducts of propositional modal logics Λ with **S5** and present new examples of product and semiproduct logics axiomatized in the 'minimal' way and enjoying the product (or semiproduct) FMP. An essential part of the proof is local tabularity of these (semi)products for Λ of finite depth; it is obtained by using bisimulation games. These results readily imply decidability for 1-variable fragments of predicate modal logics $\mathbf{Q}\Lambda$ and $\mathbf{Q}\Lambda$+Barcan formula. We also present new counterexamples, i.e. (semi)products not axiomatizable in the simplest way.

Keywords: modal logic, 1-variable fragment, product of modal logics, bisimulation game, finite model property

1 Introduction

Semiproducts and products are special types of combined modal logics. Their systematic investigation began in the 1990s, notably due to connections with other areas of logic, both pure and applied, cf. [2]. Nowadays the field has become even more interesting and intriguing; for an overview of some developments cf. [6]. In this note we are especially interested in (semi)products with **S5**, due to their interpretation in modal predicate logic translating the **S5**-necessity into the universal quantifier.

One of the starting points in the study of products was the "product-matching" theorem ([2], Theorem 5.9) — the product of two Kripke complete Horn axiomatizable logics is axiomatized in the minimal way. A similar result for semiproducts ("semiproduct-matching") is known for particular cases only (ibid., Theorem 9.10). Here we present some new positive examples — Horn axiomatizable logics that are semiproduct-matching with **S5** and have the product finite model property (FMP). This implies decidability and the FMP for corresponding 1-variable modal predicate logics.

We also present new counterexamples — two infinite families of logics not semiproduct-matching with **S5**. In particular, we show that Horn axiomatizable complete logics may not be semiproduct-matching.

2 Preliminaries

We consider normal monomodal predicate logics, as defined in [4], in a signature with predicate letters only. A logic is a set of formulas containing standard first-order axioms and the axiom of **K** and closed under standard rules (including predicate substitution). The minimal predicate extension of a propositional monomodal logic Λ is denoted by $\mathbf{Q}\Lambda$; $\mathbf{Q}\Lambda\mathbf{C}$ denotes $\mathbf{Q}\Lambda + \forall x \Box P(x) \to \Box \forall x\, P(x)$ (the Barcan axiom).

Formulas constructed from a single variable x and monadic predicate letters are called *1-variable*. Formulas in which every subformula of the form $\Box B$ contains at most one parameter are called *monodic* [2].

Lemma 2.1 *Every monadic monodic formula with at most one parameter is equivalent to a 1-variable formula in* **QK**.

In turn, every monomodal 1-variable formula A translates into a bimodal propositional formula A_* with modalities $\Box$ and $\blacksquare$, if every atom $P_i(x)$ is replaced with a proposition letter p_i and every quantifier $\forall x$ with $\blacksquare$. The *1-variable fragment* of a predicate logic L is the set

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is 1-variable}\}.$$

For a modal predicate logic L, we have the following:

Lemma 2.2 $L{-}1$ *is a bimodal propositional logic containing* $\mathbf{K} \rfloor \mathbf{S5}$.

Definition 2.3 The *product* of frames $F_1 = (U_1, R_1),\ F_2 = (U_2, R_2)$ is $F_1 \times F_2 := (U_1 \times U_2, R_h, R_v)$, where

$$R_h(u, v) = R_1(u) \times \{v\},\ R_v(u, v) = \{u\} \times R_2(v).$$

A *semiproduct* of F_1 and F_2 is a subframe $(F_1 \times F_2)|W$ where $R_h(W) \subseteq W$.

Consider a monomodal propositional logic Λ (in the language with $\Box$) and **S5** (in the language with $\blacksquare$). Put

$$\Lambda \rfloor \mathbf{S5} := \Lambda * \mathbf{S5} + \Box\blacksquare p \to \blacksquare\Box p, \quad [\Lambda, \mathbf{S5}] := \Lambda \rfloor \mathbf{S5} + \blacksquare\Box p \to \Box\blacksquare p,$$

where $*$ denotes fusion.

Definition 2.4 The *product* $\Lambda \times \mathbf{S5}$ is the logic of the class of all products of Λ-frames with **S5**-frames. Similarly, the *semiproduct* $\Lambda \leftthreetimes \mathbf{S5}$ is the logic of the class of all semiproducts of such frames.

In both cases, instead of arbitary **S5**-frames one can use single clusters.

Definition 2.5 The *Kripke-completion* $\overline{L}$ of a modal predicate logic L is the logic of the class of all predicate Kripke frames validating L.

Lemma 2.6 (i) $\Lambda \lrcorner \mathbf{S5} \subseteq \mathbf{Q}\Lambda{-}1 \subseteq \overline{\mathbf{Q}\Lambda}{-}1 = \Lambda \leftthreetimes \mathbf{S5}$.

(ii) $[\Lambda, \mathbf{S5}] \subseteq \mathbf{Q}\Lambda\mathbf{C}{-}1 \subseteq \overline{\mathbf{Q}\Lambda\mathbf{C}}{-}1 = \Lambda \times \mathbf{S5}$.

Definition 2.7 Λ and $\mathbf{S5}$ are called *semiproduct-matching* if $\Lambda \lrcorner \mathbf{S5} = \Lambda \leftthreetimes \mathbf{S5}$ and *product-matching* if $[\Lambda, \mathbf{S5}] = \Lambda \times \mathbf{S5}$.

Λ is called *quantifier-friendly*, if $\mathbf{Q}\Lambda{-}1 = \Lambda \lrcorner \mathbf{S5}$, and *Barcan-friendly*, if $\mathbf{Q}\Lambda\mathbf{C}{-}1 = \Lambda \times \mathbf{S5}$.

So Λ is quantifier-friendly (respectively, Barcan-friendly) whenever Λ and $\mathbf{S5}$ are semiproduct-matching (respectively, product-matching).

Theorem 2.8 *(cf. [2], Theorem 5.9). If Λ is Kripke complete and Horn axiomatizable, then Λ and $\mathbf{S5}$ are product-matching.*

For semiproducts an analogue of this theorem does not hold (see below). Let us recall, in a slightly more general form, a number of positive results presented in [2], Theorem 9.10. [1]

Definition 2.9 A *one-way PTC-logic* is a modal propositional logic axiomatized by formulas of the form $\Box p \to \Box^n p$ and variable-free formulas.

Theorem 2.10 *Λ and $\mathbf{S5}$ are semiproduct-matching for any one-way PTC-logic Λ.*

3 Counterexamples

Theorem 3.1 *(cf. [9]) Let*

$$\Box\mathbf{T} := \mathbf{K} + \Box(\Box p \to p), \quad \mathbf{SL4} := \mathbf{K4} + \Diamond p \leftrightarrow \Box p.$$

If $\Box\mathbf{T} \subseteq \Lambda \subseteq \mathbf{SL4}$, then Λ and $\mathbf{S5}$ are not semiproduct-matching.

For the proof note that $\Box\blacksquare(\Box p \to p) \in (\Lambda \leftthreetimes \mathbf{S5}) - (\Lambda \lrcorner \mathbf{S5})$.

Hence we obtain counterexamples to an analogue of Theorem 2.8: Horn axiomatizable logics $\Box\mathbf{T}$, $\mathbf{K5}$, $\mathbf{K45}$ are not semiproduct-matching with $\mathbf{S5}$.

Nevertheless, we have

Remark 3.2 (cf. [8]) *Every complete Horn axiomatizable logic is quantifier-friendly.*

Theorem 3.3 *If $\mathbf{K} + Alt_n \subseteq \Lambda \subseteq \mathbf{K} + Alt_n + \Box^m\bot$ for $n \geq 3$, $m \geq 2$, then Λ and $\mathbf{S5}$ are neither product- nor semiproduct-matching.*

Proof. (Sketch.) Take the product $F_1 \times F_2$, where F_1 is the irreflexive tree with the root 0 and the leaves $1, \ldots, n$ and F_2 is the two-element cluster $\{1, 2\}$; replace R_v by the least equivalence relation S_2 such $(x, y)S_2(x', y')$ for $x = x' = 0$ or $x = x' > 3$, $(1,1)S_2(2,2)$, $(1,1)S_2(3,2)$, $(1,2)S_2(2,1)$, $(1,2)S_2(3,1)$. The

[1] In [2] semiproducts are called 'expanding relativized products', $\Lambda \lrcorner \mathbf{S5}$ is denoted by $[\Lambda, \mathbf{S5}]^{EX}$, $\Lambda \leftthreetimes \mathbf{S5}$ by $(\Lambda \times \mathbf{S5})^{EX}$.

resulting frame G_n is not a p-morphic image of a semiproduct of a $(\mathbf{K}+Alt_n)$-frame and a cluster while $G_n \models [\mathbf{K}+Alt_n+\Box^2\bot, \mathbf{S5}]$. Therefore its Fine-Jankov formula belongs to $(\Lambda \leftthreetimes \mathbf{S5}) - [\Lambda, \mathbf{S5}]$. ■

A standard canonical model argument proves Kripke-completeness of all the logics $\mathbf{Q}\Lambda$ for $\Lambda = \mathbf{K} + Alt_n,\ \mathbf{K} + Alt_n + \Box^m\bot$. So we obtain

Corollary 3.4 *The logics* $\mathbf{K}+Alt_n$, $\mathbf{K}+Alt_n+\Box^m\bot$ *are not quantifier-friendly for* $n \geq 3$, $m \geq 2$.

4 Local tabularity

Recall that a propositional logic L is *locally tabular*, if for any finite k there exist finitely many L-non-equivalent formulas in k proposition letters.

It is well known that every extension of a locally tabular modal logic in the same language is locally tabular; every locally tabular logic has the FMP.

Theorem 4.1 *Every logic* $(\mathbf{K} + \Box^n\bot) \lrcorner \mathbf{S5}$ *is locally tabular.*

This theorem is proved by using bisimulation games; the corresponding technique is described in [7].

A monomodal logic Λ is *of finite depth* if $\Box^n\bot \in \Lambda$ for some n.

Corollary 4.2 *If* Λ *is of finite depth, then the logics* $\Lambda \leftthreetimes \mathbf{S5}$, $\Lambda \lrcorner \mathbf{S5}$ *have the FMP; so their finite axiomatizability implies decidability.*

In particular, $\Lambda \leftthreetimes \mathbf{S5}$ $(\Lambda \times \mathbf{S5})$ is decidable, provided Λ, $\mathbf{S5}$ are semiproduct-(product-) matching and Λ is of finite depth.

5 More examples of semiproduct-matching

In contrast with Theorem 3.3, we can identify some other logics that are semiproduct-matching with **S5**.

Lemma 5.1 *Consider the axiom* $Ath := \Diamond\Diamond p \to \Box\Diamond p$. *Ath-frames are defined by the following first-order condition:*

$$\forall x, y, z, u\, (xRy \wedge xRz \wedge yRu \to zRu).$$

We call these frames *thick*.

Proposition 5.2 *The logics* $\mathbf{K} + Ath$, $\mathbf{K} + Ath + \Box^n\bot$ *for* $n \geq 1$ *are semiproduct-matching with* **S5**.

Proof. (Sketch.) Every countable rooted $\mathbf{K} \lrcorner \mathbf{S5}$-frame H is a p-morphic image of a semiproduct G of a tree F and a cluster C; the proof is similar to the one for products, cf. [2]. Since Ath is a Horn formula, we can take the corresponding Horn closure G^+; then G^+ is a semiproduct of F^+ and C. If $H \models Ath$, we obtain a p-morphism from G^+ onto H. So every formula refutable on H is not in $(\mathbf{K} + Ath) \leftthreetimes \mathbf{S5}$.

Adding variable-free axioms $\Box^n\bot$ does not affect this argument. ■

6 Product and semiproduct FMP

In many cases (semi)products enjoy the (semi)product FMP. In particular, if L_1 is tabular and L_2 has the FMP, then $L_1 \times L_2$ has the product FMP [3, Cor. 5.9]. Probably, this may not be true, if L_1 is only locally tabular. Examples of semiproduct FMP can be found in [5], but they do not cover our next result:

Theorem 6.1 *For* $\Lambda = \mathbf{K} + Ath$ *and* $\Lambda = \mathbf{K} + \Box^n\bot + Ath$*, the (semi)product of* Λ *with* **S5** *has the (semi)product FMP.*

Corollary 6.2 *For logics* Λ *from Theorem 6.1* $\mathbf{Q}\Lambda - 1$ *has the FMP, i.e., is complete w.r.t. finite Kripke frames with finite domains.*

Let us give some comments about the proof of Theorem 6.1 for the case of semiproducts. Note that $(\mathbf{K} + Ath) \leftthreetimes \mathbf{S5} = \bigcap_n((\mathbf{K} + \Box^n\bot + Ath) \leftthreetimes \mathbf{S5})$, so it suffices to consider only $L = \Lambda \leftthreetimes \mathbf{S5}$ for $\Lambda = \mathbf{K} + \Box^n\bot + Ath$ and show that every finite rooted L-frame $F = (W, R_1, R_2)$ is a p-morphic image of a finite semiproduct of a Λ-frame with a cluster. A *row* in F is a connected component in (W, R_1); a *column* is an equivalence class under R_2; a *block* is a non-empty intersection of a row and a column. F is *straight* if all its blocks are singletons. We can show that F is a p-morphic image of a straight rooted L-frame isomorphic to a semiproduct of a Λ-frame and a cluster.

Remark 6.3 We hope our main results can be transferred to extensions of **GL**. The logic $\mathbf{GL} \rfloor \mathbf{S5}$ is the well-known provability logic of Artemov–Japaridze, which is semiproduct-matching with **S5**. A transitive analogue of *Ath* is R. Solovay's axiom $AS := \Box(\Box p \to \Box q) \vee \Box(\Box q \to p \wedge \Box p)$. We may conjecture that $\mathbf{SOL} := \mathbf{GL} + AS$ (Solovay's logic of "provability w.r.t **ZF**" cf. [1], ch. 13) is also semiproduct-matching with **S5** and that $\mathbf{SOL} \leftthreetimes \mathbf{S5}$ has the semiproduct FMP.

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