Fusions of canonical predicate modal logics are canonical

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Abstract

We prove that canonicity of predicate modal logics transfers to their fusions.

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1 Introduction

In propositional modal logic, completeness through canonicity is a powerful technique for establishing Kripke completeness. A number of general results—among them, Sahlqvist canonicity theorem—show that modal propositional logics axiomatized by formulas of particular form are canonical and, therefore, Kripke complete. Moreover, it is known that canonicity and Kripke completeness of propositional logics transfer to their fusions [1,3]. By contrast, not much is known about canonicity of predicate modal (even monomodal) logics. The authors are only aware of the following general canonicity results for predicate modal logics: the Tanaka-Ono theorem for constant domains [5], canonicity of the minimal extensions of propositional one-way PTC logics with expanding domains [2, Theorem 6.1.29], and transfer of canonicity under boxing [4, Theorem 4.1]. Neither canonicity nor completeness transfers from logics to their fusions have been studied in the predicate setting. In this brief note, we show that, in predicate logic, just as in propositional logic, canonicity transfers to fusions.

2 Preliminaries

We consider logics in two languages: the monomodal predicate language $\mathcal{L}_1$ contains countably many free variables (denoted by $a, a_1, b, \ldots$), countably many bound variables (denoted by $x, y, x_1, \ldots$),[1] countably many predicate letters of every arity, the Boolean connectives $\neg$ and $\wedge$, the quantifier symbol $\forall$, and the unary modality $\Box_1$; the bimodal language $\mathcal{L}_2$ extends $\mathcal{L}_1$ with the unary modality $\Box_2$. Formulas are defined by recursion: atomic formulas are expressions of the form $P(a_1, \ldots, a_n)$; if A and B are formulas, then so is $(A \wedge B)$; if A is a formula, then so are $\neg A$, $\Box_i A$, and $\forall x\, [x/a]A$, where $[x/a]$ is a substitution of a bound variable x not occurring in A for a free variable a. As usual, $\bot$ abbreviates $(B \wedge \neg B)$, for some fixed B. An *occurrence of a free variable a* in a formula A is a triple (A, i, a) such that a is the ith symbol of A. The universal closure of a formula A, which is unique up to the renaming of variables, is denoted by $\bar{\forall} A$.

We denote by $\Box_2 Fma$ the set of all $\mathcal{L}_2$-formulas of the form $\Box_2 A$.

An *N-modal predicate logic* (in this paper, $N \in \{1, 2\}$) is a set of $\mathcal{L}_N$-formulas containing the minimal N-modal propositional logic $\mathbf{K}_N$ and closed under Substitution (Sub), Modus Ponens (MP), Generalization (Gen), and Necessitation for $\Box_1, \ldots, \Box_N$. The *fusion* $L_1 * L_2$ of 1-modal predicate logics L_1 and L_2 is the logic $\mathbf{K}_2 + L_1 \cup L_2^{+1}$, where L_2^{+1} is obtained from L_2 by replacing every occurrence of $\Box_1$ with $\Box_2$.

We work with Kripke semantics with expanding domains (see, e.g., [2, Chapter 3]). A *predicate Kripke N-frame with expanding domains* is a tuple $\boldsymbol{F} = (F, D)$ where $F = (W, R_1, \ldots, R_N)$ is a (propositional) Kripke N-frame and $D := \{D_w \mid w \in W\}$ is a system of non-empty domains over F such that, if $i \leqslant N$ and wR_iw', then $D_w \subseteq D_{w'}$. The following fact is well known:

Fact 2.1 *Let $\boldsymbol{F}$ be a predicate Kripke 2-frame, and let L_1 and L_2 be predicate monomodal logics. If $\boldsymbol{F} \models L_1$ and $\boldsymbol{F} \models L_2$, then $\boldsymbol{F} \models L_1 * L_2$.*

We write $(W, R) \subseteq (W', R')$ if a Kripke 1-frame (W, R) is a subframe of a Kripke 1-frame (W', R'), and $(W, R) \sqsubseteq (W', R')$ if (W, R) is a generated subframe of (W', R').

For the construction of canonical models, we use languages enriched with a countable set of *constants* (denoted by $c, c_1, \ldots$). Constants behave just like free variables, except that quantified formulas cannot be obtained by replacing constants with bound variables and prefixing a quantifier. A set of sentences possibly containing constants is called a *theory*. If Γ is a theory, the set of all constants occurring in Γ is denoted by $\mathcal{C}_\Gamma$ and the set of all sentences possibly containing constants from $\mathcal{C}_\Gamma$ is denoted by $\mathcal{L}(\Gamma)$. A theory Γ is called *negation-saturated* if, for every $A \in \mathcal{L}(\Gamma)$, either $A \in \Gamma$ or $\neg A \in \Gamma$. A theory Γ is called *Henkin* if, whenever $\exists x\, A(x) \in \mathcal{L}(\Gamma)$, there exists $c \in \mathcal{C}_\Gamma$ such that $\exists x A(x) \to A(c) \in \Gamma$.

We say that a formula, possibly with constants, A is *L-provable*, and write $\vdash_L A$, if $A = [\boldsymbol{c}/\boldsymbol{a}]B$, for some $B \in L$ and some renaming $[\boldsymbol{c}/\boldsymbol{a}]$ of some free variables of B with constants.

We say that a sentence A is *L-derivable from a theory* Γ, and write $\Gamma \vdash_L A$,

[1] Note that our syntax differs from that used in [2], where there are no separate stocks of free and bound variables.

if there exists a sequence $A_1, \dots, A_n$ of formulas, called an *L-derivation of A from Γ*, such that, for every A_i, either $A_i \in \Gamma$ or $\vdash_L A_i$ or else A_i is obtained from A_j, with $j < i$, by either (MP) or (Gen). A theory Γ is *L-consistent* if $\Gamma \not\vdash_L \bot$. The following fact is well known and proven as for the classical logic:

Fact 2.2 *If Γ is a theory, a is a free variable, and c is a constant such that $c \notin \mathcal{C}_\Gamma$, then $\Gamma \vdash_L A$ implies $\Gamma \vdash_L [c/a]A$.*

Let L be an N-modal predicate logic. An *L-place* is a negation-saturated L-consistent Henkin theory Γ such that the set of constants not in $\mathcal{C}_\Gamma$ is infinite. The *canonical predicate frame* for L is the tuple $\boldsymbol{F}_L = (W_L, R_1, \dots, R_N, D_L)$ where W_L is the set of all L-places, $\Gamma R_i \Delta$ holds iff $\Box_i A \in \Gamma$ implies $A \in \Delta$, and the domain function is defined by $D_L(\Gamma) := \mathcal{C}_\Gamma$. It is well-known that every non-theorem of L is refuted on $\boldsymbol{F}_L$.

By analogy with propositional logic, we call a predicate logic *canonical* if it is validated by its canonical predicate frame. [2] Every canonical predicate logic is Kripke complete: if a predicate logic L is canonical, then $L = \{A \mid \boldsymbol{F}_L \models A\}$.

3 Main result

In view of Fact 2.1, our aim is to show that if L_1 and L_2 are canonical monomodal predicate logics, then the canonical predicate frame for $L_1 * L_2$ validates both L_1 and L_2. The arguments for L_1 and L_2 are symmetric, so we give only one in full detail.

Define a binary relation $\sim$ on *Fma* so that $A \sim B$ if B can be obtained by replacing occurrences of free variables in A with some free variables; e.g.,

$$\Box_2 \exists x (P(x,a,b) \wedge Q(x,b,c)) \;\sim\; \Box_2 \exists x (P(x,a,d) \wedge Q(x,a,b)), \qquad (*)$$

i.e, the only occurrence of a is replaced with a, the first occurrence of b with d, the second occurrence of b with a, and the only occurrence of c with b. It should be clear that $\sim$ is an equivalence; we write $[A]$ for $\{B \mid B \sim A\}$.

Enrich $\mathcal{L}_1$ with a countable set of predicate letters of each arity; denote by $\mathcal{S}$ the set of all newly introduced predicate letters and by $(\mathcal{L}_1 + \mathcal{S})$ the set of formulas of the resultant language; denote by $AF_\mathcal{S}$ the set of atomic formulas with predicate letters from $\mathcal{S}$.

Let $s\colon \Box_2 Fma/{\sim} \to \mathcal{S}$ be a bijection such that the arity of the letter $s([\Box_2 A])$ equals the number of occurrences of free variables in $\Box_2 A$; the map s is well defined since all formulas from $[\Box_2 A]$ have the same number of occurrences of free variables (e.g., four in formulas from $(*)$). We write $\Box_2 A(\boldsymbol{a})$ to mean that $\boldsymbol{a}$ is the list, with repetitions, of free variables with occurrences in $\Box_2 A$ (e.g., in the first formula from $(*)$, $\boldsymbol{a} = (a,b,b,c)$) and define a (unique) bijection $\bar{s}\colon \Box_2 Fma \to AF_\mathcal{S}$ so that $\bar{s}(\Box_2 A(\boldsymbol{a})) := s([\Box_2 A(\boldsymbol{a})])(\boldsymbol{a})$; the atomic formula $\bar{s}(\Box_2 A)$ is called the *surrogate* of $\Box_2 A$. Next, define a map $e\colon \mathcal{L}_2 \to (\mathcal{L}_1 + \mathcal{S})$

[2] Note that, in general, canonicity may depend on the cardinality of the set of constants used in the construction of a canonical predicate frame; for the purposes of this paper, however, this issue is immaterial.

by

$$\begin{array}{llll} e(A) & := A \quad \text{if } A \text{ is atomic;} & e(\forall x\,[x/a]A) & := \forall x[x/a]\,e(A); \\ e(\neg A) & := \neg e(A); & e(\Box_1 A) & := \Box_1 e(A); \\ e(A \wedge B) & := e(A) \wedge e(B); & e(\Box_2 A) & := \bar{s}(\Box_2 A). \end{array}$$

The formula $e(A) \in (\mathcal{L}_1 + \mathcal{S})$ is called the *ersatz* of $A \in \mathcal{L}_2$. If $\Gamma \subseteq \mathcal{L}_2$, then $\Gamma^e := \{e(A) \mid A \in \Gamma\}$. Note that the map e is a bijection; hence, we can define the map $r := e^{-1}$. The formula $r(A) \in \mathcal{L}_2$ is called the *reconstruction* of $A \in (\mathcal{L}_1 + \mathcal{S})$. Note that $r(A) = A$ if A does not contains predicate letters from $\mathcal{S}$. If $\Gamma \subseteq (\mathcal{L}_1 + \mathcal{S})$, then $\Gamma^r := \{r(A) \mid A \in \Gamma\}$.

Lemma 3.1 *For every $A \in (\mathcal{L}_1 + \mathcal{S})$, the formula $r(A)$ is a substitution instance of A.*

Proof. Induction on A. □

Let $L := L_1 * L_2$, and let

$$F_L := (W_L, R_L, D_L) \quad \text{and} \quad F_{L_1} := (W_{L_1}, R_{L_1}, D_{L_1})$$

be the canonical predicate frames of, respectively, L and L_1; let, also,

$$W_L^e := \{\Gamma^e \mid \Gamma \in W_L\}.$$

Lemma 3.2 *If $\Gamma \subseteq \mathcal{L}_2$ is negation-saturated and Henkin, then so is Γ^e.*

Proof. To see that Γ^e is negation-saturated, assume that $A \in \mathcal{L}(\Gamma) - \Gamma^e$. Then, $r(A) \notin \Gamma$, and so, since Γ is negation-saturated, $\neg r(A) (= r(\neg A)) \in \Gamma$. Thus, $e(r(\neg A)) (= \neg A) \in \Gamma^e$. The argument for Henkinness is similar. □

Lemma 3.3 *If $\Gamma \subseteq (\mathcal{L}_1 + \mathcal{S})$ is negation-saturated and Henkin, then so is Γ^r.*

Proof. Similar to the proof of Lemma 3.2. □

Lemma 3.4 $W_L^e \subseteq W_{L_1}$.

Proof. Let Γ be an L-place. By Lemma 3.2, Γ^e is negation-saturated and Henkin. Clearly, $\mathcal{C}_{\Gamma^e} = \mathcal{C}_\Gamma$. It remains to show that Γ^e is L_1-consistent.

Suppose not, i.e., $\Gamma^e \vdash_{L_1} \bot$. Then, there exists an L_1-derivation $A_1, A_2, \ldots, \bot$ of $\bot$ from Γ^e. Then, as we next show,

$$r(A_1), r(A_2), \ldots, r(\bot) (= \bot)$$

is an L-derivation of $\bot$ from Γ. Indeed, if $A_i \in \Gamma^e$, then $r(A_i) \in (\Gamma^e)^r (= \Gamma)$. If $\vdash_{L_1} A_i$, then there exists a renaming $[\boldsymbol{a}/\boldsymbol{c}]$ of the constants occurring in A_i into free variables such that $[\boldsymbol{a}/\boldsymbol{c}]A_i \in L_1$. Since $L_1 \subseteq L_1 * L_2 (= L)$, surely $[\boldsymbol{a}/\boldsymbol{c}]A_i \in L$; hence, $\vdash_L A_i$. Since L is closed under substitution, it follows, by Lemma 3.1, that $\vdash_L r(A_i)$. Lastly, the map r clearly commutes with both (MP) and (Gen). Thus, $\Gamma \vdash_L \bot$, contrary to L-consistency of Γ. □

If Γ is a set of formulas, we denote by $\overline{\Gamma}$ the set of sentences from Γ.

Lemma 3.5 *If* $\Gamma \in W^e_L$*, then* $\overline{L}^e \subseteq \Gamma$.

Proof. Let $\Gamma \in W^e_L$. Then, $\Gamma = \Delta^e$, for some $\Delta \in W_L$. Since, by [2, Lemma 6.1.4(1)], $\overline{L} \subseteq \Delta$, it follows that $\overline{L}^e \subseteq \Delta^e (= \Gamma)$. □

Lemma 3.6 *If* $\Gamma \in W_{L_1}$ *and* $\overline{L}^e \subseteq \Gamma$*, then* $\Gamma \in W^e_L$.

Proof. Suppose that $\overline{L}^e \subseteq \Gamma \in W_{L_1}$. We prove that $\Gamma^r \in W_L$; since $\Gamma = (\Gamma^r)^e$, it then follows that $\Gamma \in W^e_L$. By Lemma 3.3, Γ^r is negation-saturated and Henkin. Clearly, $\mathcal{C}_{\Gamma^r} = \mathcal{C}_\Gamma$. It remains to show that Γ^r is L-consistent.

Suppose not, i.e., $\Gamma^r \vdash_L \bot (= r(\bot))$. We will prove that, then, $\Gamma \vdash_{L_1} \bot$. To that end, we show that $\Delta \vdash_L A$ and $\overline{L}^e \subseteq \Delta^e$ imply $\Delta^e \vdash_{L_1} e(A)$, for every $\Delta \in W_L$ and every $A \in \mathcal{L}_2$. (The required conclusion then follows from the fact that $(\Gamma^r)^e = \Gamma$.) We proceed by induction on the L-derivation of A from Δ. If $B \in \Delta$, then $e(B) \in \Delta^e$. Suppose, next, that $\vdash_L B$. We may assume that B does not contain any constants from Δ, and hence any constants from Δ^e. Since $\vdash_L B$, there exists a renaming $[\boldsymbol{a}/\boldsymbol{c}]$ of constants into free variables such that $[\boldsymbol{a}/\boldsymbol{c}]B \in L$. Let $B' := [\boldsymbol{a}/\boldsymbol{c}]B$. By (Gen), $\overline{\forall} B' \in \overline{L}$. Since, by assumption, $\overline{L}^e \subseteq \Delta^e$, it follows that $e(\overline{\forall} B')(= \overline{\forall}\, e(B')) \in \Delta^e$. Thus, we can add $\overline{\forall}\, e(B')$ and the L_1-theorem $\overline{\forall}\, e(B') \to e(B')$ to any L_1-derivation from Δ^e; hence, $\Delta^e \vdash_{L_1} e(B')$. Now, let $[\boldsymbol{c}/\boldsymbol{a}] := [\boldsymbol{a}/\boldsymbol{c}]^{-1}$. Then, by Fact 2.2 and by our assumption about constants, $\Delta^e \vdash_{L_1} [\boldsymbol{c}/\boldsymbol{a}]e(B')(= e([\boldsymbol{c}/\boldsymbol{a}]B') = e(B))$. Lastly, the map e clearly commutes with (MP) and (Gen). Hence, $\Gamma \vdash_{L_1} \bot$, contrary to L_1-consistency of Γ. □

Let $R^e_L := R_{L_1} \upharpoonright W^e_L$.

Lemma 3.7 $(W^e_L, R^e_L) \sqsubseteq (W_{L_1}, R_{L_1})$.

Proof. It follows from Lemma 3.4 and the definition of R^e_L that $(W^e_L, R^e_L) \subseteq (W_{L_1}, R_{L_1})$. To see that $R_{L_1}(W^e_L) \subseteq W^e_L$, suppose that $\Gamma \in W^e_L$ and $\Gamma R_{L_1} \Delta$. Since $\Gamma \in W^e_L$, it follows, by Lemma 3.5, that $\overline{L}^e \subseteq \Gamma$. Since $\Box_1 \overline{L} \subseteq \overline{L}$, surely $(\Box_1 \overline{L})^e (= \Box_1 \overline{L}^e) \subseteq \Gamma$. Since $\Gamma R_{L_1} \Delta$, the definition of R_{L_1} implies that $\overline{L}^e \subseteq \Delta$. Hence, by Lemma 3.6, $\Delta \in W^e_L$. □

Proposition 3.8 *If* $\boldsymbol{F}_{L_1} \models L_1$*, then* $\boldsymbol{F}_{L_1 * L_2} \models L_1$.

Proof. Suppose that $\boldsymbol{F}_{L_1} \models L_1$. Since e is an embedding, Lemma 3.7 means that the frame $(W_{L_1 * L_2}, R_1)$ is isomorphic to a generated subframe of $\boldsymbol{F}_{L_1}$. Since validity of predicate formulas is preserved under generated subframes [2, Lemma 3.3.18], it follows that $(W_{L_1 * L_2}, R_1) \models L_1$ and hence $\boldsymbol{F}_{L_1 * L_2} \models L_1$. □

Lemma 3.9 *If* $\boldsymbol{F}_{L_2} \models L_2$*, then* $\boldsymbol{F}_{L_1 * L_2} \models L_2$.

Proof. The argument here is analogous to that of Proposition 3.8. We define surrogates of formulas of the form $\Box_1 A$ in the language $(\mathcal{L}_2 + \mathcal{S})$, and proceed as before, but swapping the roles of L_1 and L_2. □

Theorem 3.10 *Let* L_1 *and* L_2 *be predicate modal logics. Then,*

$$\boldsymbol{F}_{L_1} \models L_1 \ \&\ \boldsymbol{F}_{L_2} \models L_2 \implies \boldsymbol{F}_L \models L_1 * L_2.$$

In other words, the fusion of two canonical predicate modal logics is a canonical predicate modal logic.

Proof. Immediate from Proposition 3.8, Lemma 3.9 and Fact 2.1. □

Since an analogue of Theorem 3.10 can be proven for polymodal logics, we obtain the following:

Corollary 3.11 *Let $L_1, \ldots, L_n$ be predicate modal logics. Then,*

$$\boldsymbol{F}_{L_1} \models L_1 \ \& \ \ldots \ \& \ \boldsymbol{F}_{L_n} \models L_n \Longrightarrow \boldsymbol{F}_L \models L_1 * \ldots * L_n.$$

In other words, the fusion of any number of canonical predicate modal logics is a canonical predicate modal logic.

The previous treatment is likely to extend to logics of constant domains, with canonicity replaced by C-canonicity [2, Chapter 7]:

Conjecture 3.12 *For predicate modal logics, C-canonicity transfers to fusions.*

Also, we believe that additional techniques should enable us to prove transfer of Kripke completeness rather than simply canonicity:

Conjecture 3.13 *For predicate modal logics, strong Kripke completeness and Kripke completeness transfer to fusions.*

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