Kripke (in)completeness of predicate modal logics with axioms of bounded alternativity

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Predicate modal logics

We consider logics in the language with $\bot$, $\to$, $\forall$, and $\Box$ that are extensions of the classical predicate logic $\mathbf{QCL}$.

A *precicate modal logic* is a set of formulas containing $\mathbf{QCL}$ and the minimal normal propositional modal logic $\mathbf{K}$, and closed under Substitution, Modus Ponens, Generalization, and Necessitation.

The minimal precicate modal logic is $\mathbf{QK}$.

If Γ and Δ are sets of formulas, then $\Gamma + \Delta$ denotes the closure of $\Gamma \cup \Delta$ under Substitution, Modus Ponens, Generalization, and Necessitation.

If Λ is a propositional modal logic, then $\mathbf{Q}\Lambda := \mathbf{QK} + \Lambda$.

Kripke semantics with expanding domains

Definition

A ***predicate frame*** is a tuple $\boldsymbol{F} = (W, R, D)$, where (W, R) is a Kripke frame and $D = \{D_u \mid u \in W\}$ is a family of non-empty domains satisfying the expaning domains condition:

$$wRv \implies D_w \subseteq D_v.$$

Definition

A ***Krike model*** is a tuple $(\mathbb{F}, I)$, where $\boldsymbol{F}$ is a predicate frame and I is an interpretation function: $I_w(P) \subseteq D_w^n$, for every $w \in W$ and every n-ary predicate symbol P.

Truth relation in Kripke models

A D_w-sentence is an expression obtained from a formula by replacing free variables with copies of elements of D_w.

The truth relation $\models$ between models M, worlds w, and D_w-sentences φ is defined by recursion:

- $M, w \models P(a_1, \ldots, a_n)$ if $(a_1, \ldots, a_n) \in I_w(P)$;
- $M, w \models \neg\varphi$ if $M, w \not\models \varphi$;
- $M, w \models \varphi \wedge \psi$ if $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \forall x\, \varphi(x)$ if $M, w \models \varphi(a)$ whenever $a \in D_w$;
- $M, w \models \Box\varphi$ if $M, v \models \varphi$ whenever $v \in R(w)$.

A formula is true in a model if its universal closure is true at every world. A formula is valid on a predicate frame $\boldsymbol{F}$ if it is true in every model over $\boldsymbol{F}$. A formula is valid on a Kripke frame F if it is valid on every predicate frame over F.

The set of formulas valid on all Kripke frames coincides with **QK**.

Kripke completeness

A predicate modal logic L is *Kripke complete* if it coincides with the set of validities on a class of predicate frames, i.e., for every $\varphi \notin L$, there exists a predicate frame $\boldsymbol{F}$ such that $\boldsymbol{F} \models L$, but $\boldsymbol{F} \not\models \varphi$.

A formula φ follows from a set Γ of formulas in Kripke semantics (notation: $\Gamma \models \varphi$) if, for every predicate frame $\boldsymbol{F}$,

$$\boldsymbol{F} \models \Gamma \quad \Longrightarrow \quad \boldsymbol{F} \models \varphi.$$

Logics considered in this talk

Let $n \geqslant 1$. Define

$$\begin{aligned}
\boldsymbol{alt}_n &:= \neg \bigwedge_{0 \leqslant i \leqslant n} \Diamond (P_i \wedge \bigwedge_{j \neq i} \neg P_j); \\
\boldsymbol{ref} &:= \Box P \to P; \\
\mathbf{4} &:= \Box P \to \Box\Box P.
\end{aligned}$$

It is well known that a Kripke frame $F = (W, R)$ validates $\boldsymbol{alt}_n$ iff $|R(w)| \leqslant n$ whenever $w \in W$.

Define

$$\begin{aligned}
\mathbf{QAlt}_n &:= \mathbf{QK} + \boldsymbol{alt}_n; \\
\mathbf{QTAlt}_n &:= \mathbf{QAlt}_n + \boldsymbol{ref}; \\
\mathbf{QK4Alt}_n &:= \mathbf{QAlt}_n + \mathbf{4}; \\
\mathbf{QS4Alt}_n &:= \mathbf{QK4Alt}_n + \boldsymbol{ref}.
\end{aligned}$$

Proposition

Let $n \geqslant 1$. Then, $\mathbf{QAlt}_n$, $\mathbf{QTAlt}_n$, $\mathbf{QK4Alt}_n$, and $\mathbf{QS4Alt}_n$ are not canonical, even though the corresponding propositional logics are.

Strong Kripke completeness of $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$

Even though $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are not canonical, we have the following:

Theorem

Logics $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are strongly Kripke complete.

The outline of the proof is presented in the abstract. In a nutshell, given an L-consistent ($L \in \{\mathbf{QAlt}_n, \mathbf{QTAlt}_n\}$) theory Γ_0, we select a submodel $M = (W, R, D, I)$ of the canonical model M_L containing theory including Γ_0 and based on a predicate frame validating L in such a way that

$$\Gamma \in W \ \& \ \Diamond\varphi \in \Gamma \implies \exists\Delta \in R(\Gamma)\big(\varphi \in \Delta\big).$$

Kripke incompleteness of $\mathbf{QK4Alt}_n$ and $\mathbf{QS4Alt}_n$

On the other hand, $\mathbf{QK4Alt}_n$ and $\mathbf{QS4Alt}_n$ are Kripke incomplete.

We show that, we use the semantics of Kripke bundles.

Kripke bundles

Definition

A ***Kripke bundle*** is a tuple $\mathbb{F} = (W, R, D, \rho)$, where (W, R) is a Kripke frame, $D = \{D_u \mid u \in W\}$ is a family of non-empty disjoint domains, and $\rho = \{\rho_{uv} \mid (u, v) \in R\}$ is a family of *inheritance relations* $\rho_{uv} \subseteq D_u \times D_v$ satisfying the constraint that $\rho_{uv}(a) \neq \varnothing$ whenever uRv and $a \in D_u$ (i.e., every individual has at least one inheritor in each accessible world).

Definition

A ***Krike bundle model*** is a tuple $(\mathbb{F}, I)$, where $\mathbb{F}$ is a Kripke bundle and I is an interpretation, defined as for Kripke models, i.e., $I_w(P) \subseteq D_w^n$, for every $w \in W$ and every n-ary predicate symbol P.

Truth relation in bundles

The truth relation $\Vdash$ between models M, worlds w, and D_w-sentences φ is defined by recursion:

- $M, w \Vdash P(a_1, \ldots, a_n)$ if $(a_1, \ldots, a_n) \in I_w(P)$;
- $M, w \Vdash \neg\varphi$ if $M, w \nVdash \varphi$;
- $M, w \Vdash \varphi \wedge \psi$ if $M, w \Vdash \varphi$ and $M, w \Vdash \psi$;
- $M, w \Vdash \forall x\, \varphi(x)$ if $M, w \Vdash \varphi(a)$ whenever $a \in D_u$;
- $M, w \Vdash \Box\varphi(a_1, \ldots, a_n)$, with distinct $a_1, \ldots, a_n \in D_u$ if

$$\forall v \in R(w)\, \forall b_1 \in \rho_{uv}(a_1) \ldots \forall b_n \in \rho_{uv}(a_n)\, M, v \Vdash \varphi(b_1, \ldots, b_n).$$

NB If a occurs in $\Box\varphi$ several times, then determining whether $M, w \models \Box\varphi$ involves substituting the same inheritor for every occurrence of a in $\Box\varphi$.

Truth and validity

A formula is *true* in a Kripke bundle model if its universal closure is true at every world of the model. A formula φ is *strongly valid* in a Kripke bundle $\mathbb{F}$ (notation: $\mathbb{F} \Vdash \varphi$) if every substitution instance of φ is true in every model over $\mathbb{F}$.

Proposition

Let $\mathbb{F}$ be a Kripke bundle. Then the set $\{\varphi \mid \mathbb{F} \Vdash \varphi\}$ is a predicate modal logic.

Validating a propositional logic

Definition

With a Kripke bundle $\mathbb{F} = (W, R, D, \rho)$, we associate a family $\{(W_k, R_k) \mid k < \omega\}$ of Kripke frames, in the following way. Define

- $W_0 := W$ and $R_0 := R$;
- $W_1 := \bigcup\{D_w \mid w \in W\}$;
- $R_1 := \bigcup\{\rho_{uv} \mid (u, v) \in R\}$;
- for every $k > 1$,

$$W_k := \bigcup\{D_w^k \mid w \in W\};$$

$$R_k := \{(\mathbf{a}, \mathbf{b}) \in D_k \times D_k \mid \forall i\, a_i R_1 b_i \;\&\; \forall i, j\, (a_i = a_j \Rightarrow b_i = b_j)\}.$$

Proposition

Let $\mathbb{F}$ be a Kripke bundle and φ be a propositional modal formula. Then $\mathbb{F} \Vdash \varphi$ iff $F_k \models \varphi$, for every $k < \omega$.

Kripke incompleteness of $\mathbf{QK4Alt}_n$

Theorem

For every $n \geqslant 1$, the logic $\mathbf{QK4Alt}_n$ is Kripke incomplete.

Proof.
Let $n \geqslant 1$ and $L = \mathbf{QK4Alt}_n$. Consider the formula

$$A_n := \Diamond^{n+1}\top \to \Diamond\,\forall x\,(\Box P(x) \to P(x)).$$

We show that (1) $L \models A_n$, but (2) $A_n \notin L$.

(1) Suppose that $\boldsymbol{F} = (W, R, D)$ and $\boldsymbol{F} \models L$. Then, R is transitive and $|R(w)| \leqslant n$ whenever $w \in W$. Assume $\boldsymbol{F}, w_0 \models \Diamond^{n+1}\top$. Then, there exist $w_1, \ldots, w_{n+1} \in W$ such that $w_0 R w_1 \ldots w_n R w_{n+1}$. Since R is transitive and $|R(w_0)| \leqslant n$, there exists $i \neq j$ such that $w_i = w_j$. Since R is transitive, w_j is reflexive. Hence, $\boldsymbol{F}, w_j \models \forall x\,(\Box P(x) \to P(x))$, and so $\boldsymbol{F}, w_0 \models \Diamond\forall x\,(\Box P(x) \to P(x))$. Since w_0 was chosen arbitrarily, $L \models A_n$.

(2) To show that $A_n \notin L$, it suffices, in view of Proposition, to obtain a Kripke bundle strongly validating L, but not A_n. Define

$$W := \{w\},\ R := \{(w, w)\},\ D_w := \{a, b\},$$

and

$$\rho_{ww} := \{(a, b), (b, b)\}.$$

Put $\mathbb{F}_0 := (W, R, D, \rho)$. Then, $\mathbb{F}_0$ is a Kripke bundle. To see that $\mathbb{F}_0 \not\Vdash A_n$, consider the model $M_0 = (\mathbb{F}_0, I)$ with $I_w(P) = \{b\}$. Since w is the only world accessible from w, the individual b is the only inheritor of a, and $M_0, w \models P(b)$, it follows that $M_0, w \models \Box P(a)$. Since $M_0, w \not\models P(a)$, it follows that $M_0, w \not\models \Box P(a) \to P(a)$ and so $M_0, w \not\models \Diamond \forall x\, (\Box P(x) \to P(x))$. On the other hand, since R is serial, $M_0, w \models \Diamond^{n+1} \top$. Hence $M_0, w \not\models A_n$, and so $\mathbb{F}_0 \not\models A_n$.

Kripke incompleteness of $\mathbf{QK4Alt}_n$ (contd.)

It remains to show that $\mathbb{F}_0 \Vdash L$. Since $\mathbf{Alt}_n \subseteq \mathbf{Alt}_m$ whenever $n \geqslant m$, it is sufficient to prove that $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_1$. Consider the family $F_k = (W_k, R_k)$ of propositional frames associated with $\mathbb{F}_0$. Since $R(= R_0)$ and $\rho(= R_1)$ are both transitive and functional, $F_0 \models \mathbf{K4Alt}_1$ and $F_1 \models \mathbf{K4Alt}_1$. Let $k > 1$. Since ρ is a function with range $\{b\}$, by definition, for every $\mathbf{c}, \mathbf{e} \in D_n$,

$$\mathbf{c} R_k \mathbf{e} \quad \Longleftrightarrow \quad \forall j \, e_j = b.$$

Hence, every $\mathbf{c} \in D_k$ has exactly one R_k-successor $(b, \ldots, b) \in D_k$, which implies that R_k is transitive and functional. Hence, $F_k \models \mathbf{K4Alt}_1$, and so, by Proposition, $\mathbb{F}_0 \Vdash \mathbf{K4Alt}_1$. Also by Proposition, $\mathbb{F}_0 \Vdash \mathbf{QK}$. Hence, $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_1$.

Theorem

For every $n \geqslant 2$, the logic $\mathbf{QS4Alt}_n$ is Kripke incomplete.

Proof.

Let $n \geqslant 1$ and $L = \mathbf{QS4Alt}_n$. Consider the formula

$$B \quad := \quad \Diamond\Box\forall x\,(\Diamond\Box P(x) \to P(x)).$$

We show that $L \models B$, but $B \notin L$.

To see that $L \models B$, observe that Kripke frames for L have a final cluster, whose worlds validate $\forall x\,(\Diamond\Box P(x) \to P(x))$. Hence, Kripke frames for L validate $\Diamond\Box\forall x\,(\Diamond\Box P(x) \to P(x))$.

To see that $B \notin L$, consider the following Kripke bundle: $W := \{u\}$, $R := \{(u, u)\}$, $D_u := \{a, b\}$, $\rho_{uu} := \{(a, a), (a, b), (b, b)\}$; lastly, $\mathbb{F}_1 := (W, R, D, \rho)$. Then, $\mathbb{F}_1 \Vdash L$, but, if we put $I_w(P) = \{b\}$, then $(\mathbb{F}_1, I), w \nVdash B$.