Expressing global supervenience in inquisitive modal logic

Introduction. Inquisitive logic [4] is an approach to logic which allows us to handle in a uniform way not only formulas regimenting statements, but also formulas regimenting various kinds of questions. For example, for every formula α of propositional or predicate logic, we will also have a corresponding formula $?\alpha$ representing the yes/no question *whether* α. Adding modalities to inquisitive logics we obtain *inquisitive modal logics*, conservative extensions of standard modal logics in which modalities may be applied to questions. E.g., in the context of a standard Kripke model, we can interpret not only a modal statement of the form $\Box p$ (where p is atomic), which has the usual interpretation, but also a modal statement of the form $\Box ?p$, which expresses the fact that all successors agree on the truth value of p. This extension of $\Box$ to questions has been studied so far in the setting of propositional modal logic [6, 2]; in this context, the possibility of applying $\Box$ to questions does not add to the expressive power of standard modal logic: for instance, $\Box ?p$ is equivalent to $\Box p \vee \Box \neg p$; more generally, every modal formula of the form $\Box\mu$ where μ is a question can be turned into an equivalent formula of standard modal logic.

In this talk, we will see that the situation changes when we turn to the setting of modal *predicate* logic. In this context, there are formulas of the form $\Box\mu$, where μ is an inquisitive formula, that are not equivalent to any formula of standard modal predicate logic. Moreover, some such formulas express very interesting modal facts. In particular, we will see that by adding $\Box$ to inquisitive first-order logic, we are able to express the *global supervience* of certain properties on others, i.e., the fact that the extension of the former is functionally determined, within the given modal range, by the extension of the latter. As an example, if P and Q are unary predicates, we may say that Q globally supervenes on P at world w in case:

$$\forall v, v' \in R[w] : P_v = P_{v'} \text{ implies } Q_v = Q_{v'}$$

where $R[w] = \{v \mid wRv\}$ and P_v is the extension of P at world v (similarly for $P_{v'}, Q_v, Q_{v'}$). We will show that this property is not expressible in standard modal predicate logic, but it is expressible by a simple modal formula in inquisitive modal predicate logic. This illustrates how, in the predicate logic setting, allowing $\Box$ to apply to questions increases the expressive power of modal predicate logic in an interesting way. We then turn to the properties of the resulting modal logic. We will then show how a broad fragment of our inquisitive modal logic allows for a kind of standard translation to classical first-order logic; as a consequence, the set of validities in this fragment is recursively enumerable, and the entailment relation is compact. (This not obvious, since it is an open problem whether these properties hold for the full language of inquisitive first-order logic, even without modalities.) Interestingly, this fragment includes all modal formulas expressing global supervenience claims.

Global supervenience. Supervenience claims are at the heart of many key discussions in analytic philosophy (see [10] for an overview). The general idea behind the notion of supervenience is as follows: given two classes of properties A and B, B supervenes on A if there cannot be a difference with respect to B-properties without a corresponding difference in A-properties. This informal idea can be made precise in different ways. One understanding focuses on individuals: two individuals cannot differ in their B-properties without also differing in their A-properties; this leads to various notions of *individual* supervenience (see [8]). Another understanding focuses on worlds as a whole: two worlds cannot differ in the extension of the B-properties without also differing in the extension of the A-properties. This leads to a notion of global supervenience, which in the context of a constant-domain Kripke model can be characterized as follows:[1]

$$\begin{array}{lcl} P_1, \dots, P_n \rightsquigarrow_w Q_1, \dots, Q_m & \iff & \forall v, v' \in R[w]: \\ & & \text{if } (P_1)_v = (P_1)_{v'} \text{ and } \dots \text{ and } (P_n)_v = (P_n)_{v'} \\ & & \text{then } (Q_1)_v = (Q_1)_{v'} \text{ and } \dots \text{ and } (Q_n)_v = (Q_n)_{v'} \end{array}$$

If the above relation holds, we say that $Q_1, \dots, Q_m$ globally supervene on $P_1, \dots, P_n$ in world w. We refer to $Q_1, \dots, Q_m$ as the supervenient properties and to $P_1, \dots, P_n$ as the subvenient properties. For simplicity, we will focus on the case of a single supervenient property and a single subvenient property, in which case the relation amounts to:

$$P \rightsquigarrow_w Q \iff \forall v, v' \in R[w] : P_v = P_{v'} \text{ implies } Q_v = Q_{v'}$$

However, our discussion extends straightforwardly to the general case.

Global supervenience is not definable in modal predicate logic. Consider standard modal predicate logic, QML, interpreted over first-order Kripke models with constant domains. We claim that there is no formula α of QML such that $M, w \models \alpha \iff P \rightsquigarrow_w Q$.

To prove this, we give a model that contains a pair of worlds w_0, w_1 which agree on all formulas of QML, and yet global supervenience holds in w_1 but not in w_0. Our model has the set $\mathbb{N}$ of natural numbers as its domain. Let E be the sets of even numbers and

$$\mathcal{X} = \{X \subseteq \mathbb{N} \mid X \text{ contains finitely many even numbers and all but finitely many odd numbers}\}$$

The universe of possible worlds includes, in addition to w_0 and w_1, worlds of the form v_{Xi} where $X \in \mathcal{X}$ and $i \in \{0, 1\}$: at world v_{Xi}, the extension of P is X, while the extension of Q is either $\emptyset$ of $\mathbb{N}$ depending on the Boolean value i:

$$P_{v_{Xi}} = X \qquad Q_{v_{Xi}} = \begin{cases} \mathbb{N} & \text{if } i = 1 \\ \emptyset & \text{if } i = 0 \end{cases}$$

At worlds w_0 and w_1, both extensions are empty. Next, we define a function $n : \mathcal{X} \to \{0, 1\}$ as follows: $n(X) = 0$ if $\#(X \cap E)$ is even, and $n(X) = 1$ if $\#(X \cap E)$ is odd (note that $X \cap E$ is finite by definition of $\mathcal{X}$). The accessibility relation is then defined as follows:

- $R[w_0] = \{v_{Xi} \mid X \in \mathcal{X} \text{ and } i \in \{0, 1\}\}$;
- $R[w_1] = \{v_{Xi} \mid X \in \mathcal{X} \text{ and } i = n(X)\}$;
- $R[v] = \emptyset$ for any world v distinct from w_0, w_1.

We have $P \rightsquigarrow_{w_1} Q$ but not $P \rightsquigarrow_{w_0} Q$. To see that the supervenience holds in w_1, suppose v_{Xi} and v_{Yj} are two successors of w_1 that assign the same extension to P; then $X = Y$, and so by the definition of $R[w_1]$ we have $i = n(X) = n(Y) = j$, which implies that the extension of Q is the same in v_{Xi} as in v_{Yj}. However, we do not have $P \rightsquigarrow_{w_0} Q$: indeed,

[1] It should be noted that there are other notions of global supervenience, which are designed to apply in a setting where domains are not constant across worlds (see among others [12, 1, 9]). While most of the literature on global supervenience has focused on these other notions, the version that we identify seems to be an eminently natural way to cash out the idea of global supervenience in a constant-domain setting, and it is indeed the original characterization of global supervenience to be found in Kim's [8].

for any $X \in \mathcal{X}$, the worlds v_{X0} and v_{X1} are both successors of w_0, and they assign the same extension X to P, but they disagree on the extension of Q.

At the same time, w_0 and w_1 are indistinguishable by formulas of QML, since they are bisimilar (for the notion of bisimulation in the setting of QML, see [13]). To argue for this, think in terms of a bisimulation game between two players, Spoiler and Duplicator. We sketch a winning strategy for Duplicator in the game starting from w_0 and w_1. In the first part of the game, until Spoiler picks an element from the domain $\mathbb{N}$, Duplicator responds with the same element. Now suppose at some point Spoiler decides to pick a successor of w_0 or w_1. The only interesting case to consider is the one where Spoiler picks a world $v_{Xi} \in R[w_0]$ such that $i \neq n(X)$ (in any other case, Duplicator can respond simply by picking the same successor). In that case, Duplicator picks v_{Yi} where $Y = X \cup \{h\}$ for some even number h which is different from any number in X and from any number picked so far in the game (this is possible since $X \cap E$ is finite). Note that $\#(Y \cap E) = \#(X \cap E) + 1$ and therefore, since $i \neq n(X)$, we have $i = n(Y)$, which implies that indeed $v_{Yi} \in R[w_1]$, as required. At this point, since the worlds v_{Xi} and v_{Yi} that have been reached do not have any successors, the only thing Spoiler can do is to pick an object on either side. Suppose $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ are the tuples picked so far. If Spoiler picks a_{m+1}, Duplicator may always pick b_{m+1} in such a way that (i) $a_{m+1} \in X \iff b_{m+1} \in Y$; (ii) if $a_{m+1} = a_j$ for some $j \leq m$ then $b_{m+1} = b_j$, while if a_{m+1} is distinct from all the previous a_j, then b_{m+1} is distinct from all the previous b_j. It is possible to achieve both (i) and (ii), since $Y \in \mathcal{X}$ guarantees that both Y and $\mathbb{N} - Y$ are infinite, and so we may always pick a fresh element in either of them. The argument is analogous if Spoiler picks b_{m+1}. It is easy to check that this is indeed a winning strategy for Duplicator.[2]

Global supervenience in inquisitive modal logic. We consider a system $\mathsf{InqQML}_\Box$ of inquisitive modal logic obtained by adding a modality $\Box$ to inquisitive first-order logic [4]. The language is given by the following definition:

$$\varphi \ := \ Rx_1 \dots x_n \mid x_1 = x_2 \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \Box \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \exists x \varphi$$

As customary in inquisitive logic, we define $\neg\varphi := (\varphi \to \bot)$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\exists x \varphi := \neg\forall x \neg\varphi$ and $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$. The operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ are called *inquisitive disjunction* and *inquisitive existential quantifier* and regarded as question-forming operators. The fragment of the language without these operators can be identified with standard modal predicate logic QML, while the fragment including $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ but not $\exists$ will be denoted $\mathsf{InqQML}_\Box^-$.

Models for $\mathsf{InqQML}_\Box$ are standard Kripke models with constant domains. However, following inquisitive semantics [7], the interpretation of $\mathsf{InqQML}_\Box$ takes the form of a recursive definition of a relation $s \models_g \varphi$ called *support*, that holds between a set of worlds s, called an *information state*, and a formula φ (relative to an assignment g). A relation of truth at a world is retrieved by defining $w \models_g \varphi$ as a shorthand for $\{w\} \models_g \varphi$. A formula φ is said to be *truth-conditional* if support for φ boils down to truth at each world, in the sense that for every model M, state s, and assignment g:

$$s \models_g \varphi \iff \forall w \in s : w \models_g \varphi$$

Thus, if a formula is truth conditional then its semantics is completely determined by its truth conditions. The semantic clauses for atoms, connectives, and quantifiers are the standard ones from inquisitive logic (see [4]). As for formulas of the form $\Box\varphi$, they are

[2]The idea for this proof was developed in collaboration with Gianluca Grilletti.

stipulated to be truth-conditional with the following truth conditions:

$$w \models_g \Box\varphi \iff R[w] \models_g \varphi$$

It is easy to check that every formula α of standard modal logic (i.e., without $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ or $\exists$) is truth-conditional, and its truth conditions coincide with the ones given by standard Kripke semantics. On the other hand, formulas involving inquisitive connectives are not in general truth-conditional. In particular, the formula $\forall x?Px$ is supported in s just in case the extension of P is the same in all worlds in s:

$$s \models \forall x?Px \iff \forall v, v' \in s : P_v = P_{v'}$$

Using this fact and the semantic clauses, it is easy to check that the modal formula $\Box(\forall x?Px \to \forall x?Qx)$ is true at a world just in case Q globally supervenes on P:

$$w \models \Box(\forall x?Px \to \forall x?Qx) \iff P \rightsquigarrow_w Q$$

Thus, in $\mathsf{InqQML}_\Box$ we have a (truth-conditional) formula that expresses the global supervenience of Q on P. Given the results in the previous section, this implies that the formula $\Box(\forall x?Px \to \forall x?Qx)$ is not equivalent to any formula of QML. Thus, in contrast to the propositional case, in the predicate logic setting allowing $\Box$ to apply to questions allows us to express (interesting) modal properties that standard modal logic cannot express.

More generally, the claim that $Q_1, \ldots, Q_m$ supervene on $P_1, \ldots, P_n$ is expressed by:

$$\Box(\forall x?P_1x \wedge \cdots \wedge \forall x?P_nx \;\to\; \forall x?Q_1x \wedge \cdots \wedge \forall x?Q_mx)$$

Two remarks: first, formulas expressing global supervenience relations do not contain $\exists$, and so they are in the fragment $\mathsf{InqQML}_\Box^-$. Second, such formulas take the form of strict conditionals $\Box(\varphi \to \psi)$ whose antecedent and consequent are questions; more specifically, the antecedent is the question asking for the extension of the subvenient properties, while the consequent is the question asking for the extension of the supervenient properties.[3]

Meta-theoretic properties of $\mathsf{InqQML}_\Box^-$. We will show that $\mathsf{InqQML}_\Box^-$, the $\exists$-free fragment of our logic, retains the key meta-theoretic properties of first-order logic. To show this, we build on ideas developed in [11] and [5]. First, we show that we can inductively define for each formula φ of $\mathsf{InqQML}_\Box^-$ a number n_φ such that φ is n_φ*-coherent* in the sense that the following holds for any M, s, g:

$$M, s \models_g \varphi \iff \forall t \subseteq s \text{ with } \#t \leq n_\varphi : M, t \models_g \varphi$$

Second, we define a family of translations from $\mathsf{InqQML}_\Box^-$ to a suitable two-sorted first-order language equipped with world variables. In particular, for any finite set $\mathsf{s} = \{\mathsf{w}_1, \ldots, \mathsf{w}_n\}$ of world variables, we define a corresponding translation $\mathrm{tr}_\mathsf{s}(\varphi)$ which behaves semantically like φ on information states of size up to the number n of world variables in s. We then show that for any set $\Phi \cup \{\psi\}$ of formulas from $\mathsf{InqQML}_\Box^-$, if s is a set consisting of n_ψ-many world variables, we have:

$$\Phi \models_{\mathsf{InqQML}_\Box^-} \psi \iff \mathrm{tr}_\mathsf{s}(\Phi) \models_{\mathsf{FOL}} \mathrm{tr}_\mathsf{s}(\psi)$$

where on the left we have entailment in $\mathsf{InqQML}_\Box^-$ and on the right entailment in classical (two-sorted) first-order logic. Using this connection, it is then easy to show that $\mathsf{InqQML}_\Box^-$ is *entailment-compact* (i.e., $\Phi \models \psi$ implies $\Phi_0 \models \psi$ for some finite $\Phi_0 \subseteq \Phi$) and the set of theorems of $\mathsf{InqQML}_\Box^-$ is recursively enumerable. In sum, we will argue that $\mathsf{InqQML}_\Box^-$ provides a natural extension of standard modal predicate logic that, among other things, allows us to regiment reasoning about global supervenience claims.

[3] In the setting of propositional modal logic, inquisitive strict conditionals have been studied in [3].

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