An Andersonian-Kangerian Reduction of Term-Modal S5

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Term-modal logics (**TMLs**) are highly expressive first-order modal formalisms. They combine a full first-order language with modal operators indexed with terms (i.e. variables or constants) of the language. We denote these term-modal operators as $\Box_\theta$. The addition of such operators allows one to express complex sentences such as 'everyone believes that they are the hero of their own story' ($\forall x(\Box_x Hx)$). It was recently proven that many term-modal logics that do not validate the T-axiom are in fact fragments of standard (that is, not term-modal) first-order modal logics. It remained an open question whether the same could be proven for term-modal logics validating the T-axiom. In this talk we will partially answer this question by proposing a non term-modal logic **AKH** (based on some ideas taken from hybrid logic) of which the term-modal version of S5, **TMS5**, is a fragment.

Term-modal logics were first introduced by Fitting et al. [2] (for an overview, see [6]). We will present a simplified version of their semantics for **TMS5**, using constant instead of increasing domains. A **TMS5**-model is a tuple $M = \langle W, \mathcal{A}, R, I\rangle$, where W is a non-empty set of worlds and $\mathcal{A}$ is a non-empty set of agents. $R \subseteq W \times \mathcal{A} \times W$ is a ternary accessibility relation such that for every $w, w', w'' \in W$ and $p \in \mathcal{A}$: (1) $\langle w, p, w\rangle \in R$, (2) if $\langle w, p, w'\rangle \in R$, then $\langle w', p, w\rangle \in R$, and (3) if $\langle w, p, w'\rangle, \langle w', p, w''\rangle \in R$, then $\langle w, p, w''\rangle \in R$. Finally, I is an interpretation function assigning an element of $\mathcal{A}$ to every term and an element of $\wp(\mathcal{A}^n)$ to every pair consisting of an n-ary predicate and world $w \in W$. The semantic clauses are as usual, except that where θ is a term of the language, $M, w \models \Box_\theta\varphi$ iff $M, w' \models \varphi$ for all w' such that $\langle w, I(\theta), w'\rangle \in R$. See [3, 4] or below for more details.

Fitting et al. originally had an epistemic or doxastic reading of the term-modal operators in mind. Thus, $\Box_\theta\varphi$ is to be read as 'θ knows that φ' or 'θ believes that φ'. However, given the appropriate conditions on the accessibility relation R, there is nothing stopping us from giving the term-modal operator other readings. For example, in term-modal deontic logic (**TMDL**), $\Box_\theta\varphi$ is read as the personal obligation 'φ is obligatory for θ' [3, 4]. It is in this deontic context that the question arose whether **TMLs** can be reduced to standard (not term-modal) first-order modal logics.

It is a well-known result in deontic logic that many propositional deontic logics are reducible to, i.e. are fragments of, alethic modal logics. This is known as the Andersonian-Kangerian reduction of deontic logic. Anderson and Kanger proposed systems of alethic modal logic with a normative constant G, which can be read as 'what morality prescribes', 'a sanction is not applicable', or 'this is not a bad state of affairs'. They then defined $\mathsf{O}\varphi$, 'it is obligatory that φ', as $\Box(G \rightarrow \varphi)$:

'it is necessary for what morality prescribes that φ'. It has been proven that, for example, standard deontic logic (**SDL**) is a fragment of the Andersonian-Kangerian logic **K** extended with the axiom $\Diamond G$. In other words, one can define a translation from formulas of **SDL** to formulas of the Andersonian-Kangerian logic such that for every formula φ of **SDL**, φ is **SDL**-valid iff the translation of φ is valid in the Andersonian-Kangerian logic [3].

In [3] and forthcoming work, similar reduction results have been proven for a number of term-modal (deontic) logics that do not validate the T-scheme. The logics of which the term-modal logics are a fragment have some noteworthy properties. First, they do not contain a term-modal operator, but a standard modal operator. Secondly, these logics do not have propositional, but instead predicative constants in their language. For example, where $\mathcal{Q}$ is such a constant, $\mathcal{Q}\theta$ can be read as 'θ is a good person'. The formula $\Box_\theta\varphi$, 'it is obligatory for θ that φ', is then defined as $\Box(\mathcal{Q}\theta \to \varphi)$: 'it is necessary for θ being a good person that φ'. The proof for the reduction is significantly more complex than in the propositional case, but still follows the same basic outline.

Unfortunately, this approach does not seem generalizable to term-modal logics that validate the T-axiom, of which **TMS5** is an example. In this talk we propose a new logic, **AKH**, to solve this open problem. **AKH** combines the ideas of Andersonian-Kangerian logics with some ideas from hybrid logic.

The language of **AKH** extends that of first-order logic with a universal modal operator $[\mathsf{U}]$, a set of set variables $SV = \{X, Y, \ldots\}$ and a $\downarrow$ binder.[1] More precisely, the language is defined as follows. Let $C = \{a, b, \ldots\}$ be the set of constants and $V = \{x, y, \ldots\}$ be the set of variables. We let ν range over V. Let $T = C \cup V$ be the set of terms (always denoting persons) and $\theta, \theta_1, \ldots$ the metavariables ranging over it. For each natural number n we let $\mathcal{P}^n$ be a set of n-ary predicate symbols and we let $\mathcal{P}$ be the union of all $\mathcal{P}^n$. We let P range over $\mathcal{P}$. Let $SV = \{X, Y, \ldots\}$ be the set of set variables and let $\mathcal{X}$ be the meta-variable ranging over this set. Lastly, we let φ, ψ, χ be metavariables for formulas. Our language $\mathcal{L}$ is defined by the following Backus-Naur form:

$$\varphi ::= P\theta_1 \ldots \theta_n \mid \varphi \vee \varphi \mid \neg\varphi \mid [\mathsf{U}]\varphi \mid \forall\nu(\varphi) \mid \mathcal{X}^\theta \mid \downarrow\mathcal{X}^\theta(\varphi)$$

The semantics of **AKH** are given by the following definitions.

Definition 1 (Models)**.** *An* **AKH**-*model is a tuple $M = \langle W, \mathcal{A}, f, I\rangle$ such that:*

1 $W \neq \emptyset$ is the world-domain and $\mathcal{A} \neq \emptyset$ is the agent-domain

2 $f : \mathcal{A} \to \wp(\wp(W))$ is a function such that for every $p \in \mathcal{A}$, $f(p)$ is a partition of W, i.e. (1) for all distinct $\Gamma, \Delta \in f(p)$, $\Gamma \cap \Delta = \emptyset$, and (2) $\bigcup f(p) = W$

3 I is an interpretation *function that assigns to every $\theta \in T$ a $p \in \mathcal{A}$ and to every pair $\langle P, w\rangle \in \mathcal{P}^n \times W$ an element of $\wp(\mathcal{A}^n)$ for every natural number*

[1]Note that the set variables $X, Y, \ldots$ are not predicate variables. As the semantics will show, their interpretation is more like that of the state variables in hybrid logic.

$n \in \mathbb{N}$

Definition 2 (Assignment function)**.** *An assignment function $g : SV \times \mathcal{A} \to \wp(W)$ on an* **AKH***-model $M = \langle W, \mathcal{A}, f, I\rangle$ is a function such that for every pair $\langle \mathcal{X}, p\rangle \in SV \times \mathcal{A}$, $g(\mathcal{X}, p) \in f(p)$.*

Definition 3 (ν-alternative)**.** *For any $\nu \in V$, $M' = \langle W, \mathcal{A}, f, I'\rangle$ is a ν-alternative to $M = \langle W, \mathcal{A}, f, I\rangle$ iff I' differs at most from I in the member of $\mathcal{A}$ that I' assigns to ν.*

Definition 4 ($\mathcal{X}, p$-alternative)**.** *$g_w^{\mathcal{X},p}$, the $\mathcal{X}, p$-alternative for g at w, is the function defined by letting $g_w^{\mathcal{X},p}(\mathcal{X}, p)$ be the unique $\Gamma \in f(p)$ such that $w \in \Gamma$ and letting $g_w^{\mathcal{X},p}(\mathcal{Y}, p') = g(\mathcal{Y}, p')$ for all $(\mathcal{Y}, p') \neq (\mathcal{X}, p)$.*

Definition 5 (Semantic clauses)**.** *Let $M = \langle W, \mathcal{A}, f, I\rangle$ be a model and let g be an assignment on M. Then we define:*

SC1 $M, g, w \models P\theta_1 \dots \theta_n$ *iff* $\langle I(\theta_1), \dots, I(\theta_n)\rangle \in I(P, w)$

SC2 $M, g, w \models \neg\varphi$ *iff* $M, g, w \not\models \varphi$

SC3 $M, g, w \models \varphi \vee \psi$ *iff* $M, g, w \models \varphi$ *or* $M, g, w \models \psi$

SC4 $M, g, w \models \theta = \kappa$ *iff* $I(\theta) = I(\kappa)$

SC5 $M, g, w \models [\mathsf{U}]\varphi$ *iff* $M, g, w' \models \varphi$ *for all* $w' \in W$

SC6 $M, g, w \models (\forall\nu)\varphi$ *iff for every ν-alternative M':* $M', g, w \models \varphi$

SC7 $M, g, w \models \mathcal{X}^\theta$ *iff* $w \in g(\mathcal{X}, I(\theta))$

SC8 $M, g, w \models \downarrow\mathcal{X}^\theta(\varphi)$ *iff* $M, g_w^{\mathcal{X},I(\theta)}, w \models \varphi$

The main idea behind the semantics is that for every $p \in \mathcal{A}$, the function f gives a partition of the world-domain, which corresponds with the partition induced by the accessibility relation in **TMS5**. Note that because of Definition 2, our set variables are different from the state variables usually employed in hybrid logic. In standard hybrid logic (see e.g. [1, p. 825]), every state variable is true at exactly one world, i.e. every state variable 'names' a world. In contrast, in our approach there is a partition of the set of worlds for every agent θ, and every X^θ 'names' a cell of this partition, i.e. X^θ is true in all and only the worlds in one cell of the partition. The intuition behind the $\downarrow$-operator is similar to that in hybrid logic. In hybrid logic the $\downarrow$-operator allows one to 'name' or reference the world (hence the original name *reference pointer*) [5, 1]. In **AKH**, the $\downarrow$-operator allows one to 'name' or reference the set of which the world is a part.

With this toolbox we can define the term-modal operator $\Box_\theta$ as:

$$\Box_\theta\varphi := \downarrow X^\theta([\mathsf{U}](X^\theta \to \varphi))$$

This definition is in the first place meant to be a technical definition to allow for the reduction. However, we can also give it a more intuitive reading. To do so, we stick to the epistemic reading of $\Box_\theta\varphi$, 'θ knows that φ'. Now several different readings of X^θ are possible. We can read X^θ as 'the total body of evidence of agent θ is called X'. Then 'θ knows that φ', $\Box_\theta\varphi$, is analyzed as 'if we call the total body of evidence that θ has (in this world) X, then every world where θ

has exactly this body of evidence X makes φ true'. Shortened this becomes: 'θ's total body of evidence necessarily implies φ'. Alternatively, we could read X^θ as 'all that θ knows is X' or 'θ's knowledge base is called X'. There is a fruitful philosophical discussion to be had about the proper reading of X^θ.

The reduction proposed in this talk has other upshots as well. Firstly, **AKH** is more expressive than **TMS5**. For example, the **AKH**-formulas $\downarrow X^\theta([\mathsf{U}](X^\theta \leftrightarrow \varphi))$ and $\downarrow X^\theta([\mathsf{U}](\varphi \rightarrow X^\theta))$ do not have a counterpart in **TMS5**, but do formalise useful statements. Given the reading proposed above, the first formula formalises the statement 'θ's total body of evidence is (necessarily) equivalent to φ' and the second formalises 'φ necessarily implies θ's total body of evidence' (see also [7]). Secondly, the reduction of **TMDLs** showed that term-modal logics are fragments of first-order modal logic, and thus not as exotic as they might seem at first. The fact that this simple reduction does not seem to work for **TMS5** was surprising. Perhaps even more surprising is the fact that we seem to need a highly unorthodox logic like **AKH** to reduce **TMS5** to a non term-modal logic. This deserves further investigation. Other possible paths of further research are reductions for **TMLs** with variable domain semantics, or for **TMLs** that validate the T-axiom but are weaker than **TMS5**.

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