THE PREDICATE MODAL LOGIC OF FORCING

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ABSTRACT. We report on research in progress, supervised by Joel David Hamkins and Benedikt Löwe. Hamkins and Löwe determined the (propositional) modal logic of forcing to be $\mathsf{S4.2}$; we aim to determine the predicate modal logic of forcing.

Forcing is a fundamental technique in set theory that was introduced in 1963 by Paul Cohen and was first used to prove the independence of the Continuum Hypothesis in [1]. It is a method for constructing new models of set theory by extending an already known model, the ground model, in a carefully chosen way as to allow for a considerable amount of control over the structure and truths of the extension model. The technique has revolutionized the field of set theory, leading to far-reaching applications and an abundance of new models of ZFC.

This relation between a ground model and its forcing extensions has led to the notion of the set-theoretic multiverse, a rich and complex hierarchy of set-theoretic universes. Its structure has been studied by means of a forcing interpretation of the modalities $\Box$ and $\Diamond$. For a model $\mathcal{M}$ of set theory we interpret $\mathcal{M} \models \Box\varphi$ as "in every forcing extension φ holds" and $\mathcal{M} \models \Diamond\varphi$ as "in some forcing extension φ holds". Further, we say that $\psi(p_0, ..., p_n)$ is a ZFC-provable propositional modal principle of forcing if it is a propositional modal sentence such that $\psi(\varphi_0, ..., \varphi_n)$ is provable for all set-theoretic sentences $\varphi_0, ..., \varphi_n$. The forcing interpretation of $\Box$ and $\Diamond$ was first introduced by Hamkins in [11], where the relative consistency of ZFC together with the maximality principle $\Diamond\Box p \to \Box p$ was shown. Subsequently, in [12], a new area of research, the modal logic of forcing, was introduced by Hamkins and Löwe, and the propositional modal principles of forcing that are provable from ZFC were determined to precisely match the modal logic $\mathsf{S4.2}$. This was followed by several works by various authors which further established the modal account on forcing, among them [11, 18, 16, 8, 9, 19, 21, 7, 6, 13, 10, 14, 23, 20]. The techniques developed for the study of modal logics of multiverses have been fruitfully used in other structural areas of mathematics, cf., e.g., [2, 15, 22, 3].

In the presentation, I shall report on an ongoing project to extend the results by Hamkins & Löwe to determine the *predicate modal logic of forcing*. More specifically, let $\mathcal{L}^\Diamond$ be the first-order modal language

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containing symbols for infinitely many predicates P_i of each arity and infinitely many variables $x, y, z, ...$, and let formulas of $\mathcal{L}^{\Diamond}$ be closed under Boolean connectives, modal operators and quantifiers. Where $\mathcal{L}^{\in}$ is the language of set theory, we can now define what it means to be a forcing translation.

Definition. A forcing translation is a function σ, mapping formulas ψ of $\mathcal{L}^{\Diamond}$ to formulas ψ^{σ} of $\mathcal{L}^{\in}$, defined recursively as follows, where the φ_i are $\mathcal{L}^{\in}$ formulas with as many free variables as the arity of the respective predicates P_i.

$$\begin{aligned} P_i(\bar{x})^{\sigma} &\equiv \varphi_i(\bar{x}) \\ (\psi_0(\bar{x}) \wedge \psi_1(\bar{y}))^{\sigma} &\equiv \psi_0(\bar{x})^{\sigma} \wedge \psi_1(\bar{y})^{\sigma} \\ (\neg\psi(\bar{x}))^{\sigma} &\equiv \neg\psi(\bar{x})^{\sigma} \\ (\forall x \psi(x, \bar{y}))^{\sigma} &\equiv \forall x \; \psi(x, \bar{y})^{\sigma} \\ (\Box\psi(\bar{x}))^{\sigma} &\equiv \text{ in every forcing extension } \psi(\bar{x})^{\sigma} \end{aligned}$$

In other words, σ is a forcing translation if it maps ψ to a substitution instance of ψ where predicates P_i are replaced by formulas φ_i having the same number of arguments such that each instance of φ_i takes the free variables that P_i would have taken in the same instance in the original formula.

Definition. A predicate modal assertion ψ is a ZFC-*provable principle of forcing* if for all forcing translations σ, $\mathsf{ZFC} \vdash \psi^{\sigma}$.

The goal of our project is to determine the ZFC-provable predicate principles of forcing. An example of such a principle is the *converse Barcan formula*

$$\Box\forall x P(x) \to \forall x \Box P(x),$$

which is always valid in a Kripke model with inflationary domains. Indeed, if $\forall x \varphi(x)$ is true in every forcing extension, then in particular, $\varphi(x)$ will be true in every forcing extension for every set x in the ground model, precisely because x continues to exist in the extension. In fact, this formula is even provable from the axioms and rules of first-order logic together with those of the smallest normal modal logic K (hence is included in $\mathsf{QS4.2}$ below).

It turns out that the answer to our main question might differ considerably depending on whether or not our language contains equality: we conjecture the answer to be different for the cases *without* and *with* equality.

Definition. We let $\mathsf{QS4.2}$ be the smallest set of formulas containing

(1) axioms of first-order logic without equality and
(2) $\psi(\chi_0, ..., \chi_{n-1})$ whenever $\psi(p_0, ..., p_{n-1})$ is a formula of propositional $\mathsf{S4.2}$, where the χ_i are formulas of $\mathcal{L}^{\Diamond}$ without equality,

and closed under the rules Modus Ponens, Necessitation, Universal Instantiation and Universal Generalisation. Further, we let $\mathsf{QS4.2}^=$ be defined as above but include equality in both points (1) and (2).

Conjecture 1. The ZFC-provable principles of forcing *without equality* are exactly those sentences in $\mathsf{QS4.2}$.

Conjecture 2. The ZFC-provable principles of forcing *with equality* are exactly those sentences in the smallest set of formulas containing

(1) the formulas in $\mathsf{QS4.2}^=$,

(2) *Necessary Identity* (NI)

$$\forall x \forall y (x = y \iff \Box x = y),$$

(3) *Necessary Non-identity* (NNI)

$$\forall x \forall y (x \neq y \iff \Box x \neq y),$$

(4) and *Infinite Domains* (InfD), which is the set of sentences

$$\{\exists x_0 ... \exists x_n \bigwedge_{i \neq j} x_i \neq x_j \mid n \in \omega\},$$

and is closed under the rules Modus Ponens, Necessitation, Universal Instantiation and Universal Generalisation.

The approach we aim to follow in proving these conjectures is based on the method developed in [12] and further specified in [14]. As in the propositional case, the lower bounds (i.e., showing that every formula conjectured to be a provable principle is a provable principle) are easy to verify and we can readily do so. The upper bounds (i.e., showing that no other formulas are provable principles) are considerably harder. In the propositional case, this is done by so-called *control statements* that we can determine for modal logics that are characterised by a class of finite frames (cf. [14, § 4]). Unfortunately, neither of the conjectured predicate modal logics have the finite frame property, so we must adjust this idea.

In the talk, I shall give some details of the techniques that we plan to employ to solve this technical problem and approach the proof of the two conjectures. This talk reports on work supervised by Joel David Hamkins and Benedikt Löwe.

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