Satisfiability Problem for the Bundled Fragments of First Order Modal Logic (Extended Abstract)

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In first-order modal logic (FOML), it is well-known that finding decidable fragments of FOML is hard. There are a very few fragments like the *monodic fragments* ([WZ01]) that are decidable. [1] When we bundle quantifiers and modalities together (as in $\exists x\Box$, $\Diamond\forall x$, etc.), we get new logical operators whose combinations produce interesting fragments of FOML without any restriction on the arity of predicates, the number of variables, or the modal scope. It has been shown that when the existential quantifier and a box modality were always bundled together to appear as a single quantifier-modality pair ($\exists x\Box$), the resulting fragment of FOML is decidable ([Wan17]). This fragment is motivated by epistemic operators that go beyond the classical know-that, and captures the logic of many *knowing-wh* expressions such as *knowing what, knowing how, knowing why*, and so on, e.g., *knowing how to achieve* φ is rendered as *there exists* a method x such that the agent *knows that* x can guarantee φ ([Wan18].

The motivation for 'bundling' is to restrict the occurrences of quantifiers using modalities. For instance, allowing only formulas of the form $\forall x\Box\alpha$ is one such bundling. On the other hand, we could also have $\Diamond\exists y\alpha$. Thus, there are many ways to 'bundle' the quantifiers and modalities. We call these the 'bundled operators/modalities'. The following syntax defines all possible bundled operators of one quantifier and one modality. Note that we exclude equality, constants, and function symbols from the syntax.

Definition 1 (Bundled-FOML syntax)**.** *Given a countable set of predicates $\mathcal{P}$ and a countable set of variables Var, the bundled fragment of* FOML *is the set*

[1] Monodic fragment requires that there be at most one free variable in the scope of any modal subformula.

of all formulas constructed by the following syntax:

$$\alpha ::= P(x_1, \ldots, x_n) \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box\alpha \mid \forall x\Box\alpha \mid \exists x\Box\alpha \mid \Box\forall x\alpha \mid \Box\exists x\alpha$$

where $P \in \mathcal{P}$ has arity n and $x, x_1, \ldots, x_n \in$ Var.

We denote AB (to mean forAll-Box) to be the language that allows only atomic formulas, negation, conjunction, $\Box\alpha$ and $\forall x\Box\alpha$ (dually $\exists x\Diamond\alpha$) formulas. Similarly, we have EB(Exists-Box), BA(Box-forAll) and BE(Box-Exists) to mean the fragments that allows formulas of the form $\exists x\Box\alpha$, $\Box\forall x\alpha$ and $\Box\exists x\alpha$ and their duals respectively.

Definition 2 (FOML structure)**.** *An increasing domain model for* FOML *is a tuple $\mathcal{M} = (\mathcal{W}, \mathcal{D}, \delta, \mathcal{R}, \rho)$ where $\mathcal{W}$ is a non-empty countable set called* worlds*; $\mathcal{D}$ is a non-empty countable set called* domain*; $\mathcal{R} \subseteq (\mathcal{W} \times \mathcal{W})$ is the* accessibility relation*. The map $\delta : \mathcal{W} \mapsto 2^{\mathcal{D}}$ assigns to each $w \in \mathcal{W}$ a non-empty local domain set such that whenever $(w, v) \in \mathcal{R}$ we have $\delta(w) \subseteq \delta(v)$ and $\rho : (\mathcal{W} \times \mathcal{P}) \mapsto \bigcup_n 2^{\mathcal{D}^n}$ is the* valuation function*, which specifies the interpretation of predicates at every world over the local domain with appropriate arity. The model $\mathcal{M}$ is said to be a* constant domain model *if for all $w \in \mathcal{W}$ we have $\delta(w) = \mathcal{D}$. When $\delta(w) = \delta(v)$ for all $w, v \in \mathcal{W}$, we call $\mathcal{M}$ is a constant domain model.*

Definition 3 (FOML semantics)**.** *Given an* FOML *model $\mathcal{M} = (\mathcal{W}, \mathcal{D}, \delta, \mathcal{R}, \rho)$ and $w \in \mathcal{W}$, and σ relevant at w, for all* FOML *formulas α define $\mathcal{M}, w, \sigma \models \alpha$ inductively as follows:*

$\mathcal{M}, w, \sigma \models P(x_1, \ldots, x_n)$	$\Leftrightarrow$	$(\sigma(x_1), \ldots, \sigma(x_n)) \in \rho(w, P)$
$\mathcal{M}, w, \sigma \models \neg\alpha$	$\Leftrightarrow$	$\mathcal{M}, w, \sigma \not\models \alpha$
$\mathcal{M}, w, \sigma \models \alpha \wedge \beta$	$\Leftrightarrow$	$\mathcal{M}, w, \sigma \models \alpha$ *and* $\mathcal{M}, w, \sigma \models \beta$
$\mathcal{M}, w, \sigma \models \exists x\alpha$	$\Leftrightarrow$	*there is some $d \in \delta(w)$ such that $\mathcal{M}, w, \sigma_{[x \mapsto d]} \models \alpha$*
$\mathcal{M}, w, \sigma \models \Box\alpha$	$\Leftrightarrow$	*for every $u \in \mathcal{W}$ if $(w, u) \in \mathcal{R}$ then $\mathcal{M}, u, \sigma \models \alpha$*

Note that in bundled fragments such as EB, a modality comes right after a quantifier as in $\exists x\Box\varphi$, thus $\exists x(\Box\varphi \wedge \Diamond\psi)$ is not in the fragment EB whatever φ and ψ are. We may weaken this condition to allow formulas of the form $\exists x\beta$ where β is a boolean combination of atomic formulas and modal formulas. Moreover, we can allow a quantifier alternation of the form $\exists x_1 \cdots \exists x_n\ \forall y_1 \cdots \forall y_m\ \beta$. As a result, we obtain *loosely bundled fragment* (LBF):

Domain	$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Upper/ Lower Bound
Constant	✓	$\star$	$\star$	$\star$	Undecidable
	$\star$	$\star$	✓	$\star$	
	✗	✓	✗	✗	PSPACE-complete
	✗	✗	✗	✓	No FMP
	✗	✓	✗	✓	
Increasing	✓	✗	✗	✗	PSPACE-complete
	✗	✓	✗	✗	
	✗	✗	✓	✗	
	✗	✗	✗	✓	EXPSPACE/ PSPACE
	✓	✓	✗	✗	EXPSPACE/NEXPTIME
	✗	✗	✓	✓	
	$\star$	✓	✓	$\star$	Undecidable
	✗	✓	✗	✓	No FMP
	✓	✓	✗	✓	Undecidable
	✓	✗	✓	✓	EXPSPACE/ NEXPTIME
	loosely bundled				

Figure 1: Satisfiability problem classification for combinations of bundled fragments. ($\star$ means that the result holds with or without the presence of the corresponding bundle.)

Definition 4 (LBF syntax)**.** *The loosely bundled fragment of* FOML *is the set of all formulas constructed by the following syntax of α:*

$$\begin{aligned}\psi &::= \quad P(z_1,\dots z_n) \mid \neg P(z_1,\dots z_n) \mid \psi\wedge\psi \mid \psi\vee\psi \mid \Box\alpha \mid \Diamond\alpha\\ \alpha &::= \quad \psi \mid \alpha\wedge\alpha \mid \alpha\vee\alpha \mid \exists x_1\dots\exists x_k\forall y_1\dots\forall y_l\ \psi\end{aligned}$$

where $k,l,n \geq 0$, $P\in\mathcal{P}$ has arity n and $x_1,\dots x_k$, $y_1,\dots y_l$, $z_1,\dots,z_n\in$ Var.

Besides EB, the bundled fragment ABEB is still decidable over increasing domain models, though it was later shown that there was a price to be paid in terms of complexity ([PRW18]). This opens up a range of questions: what about other bundles, such as BE or BA and combinations thereof? Which of these distinguishes constant domain and increasing domain models? What about further bundles such as $\forall x\exists y\Box$ etc.? Can we identify the borderline between decidability and undecidability in this terrain? In [LPRW22] we consider all the bundles and classify them as: decidable ones, undecidable ones, and for those without definite answers yet, we show they lack the finite model property. Moreover, the LBF generalizes the bundling idea to what we believe to be the strongest yet decidable bundled fragment. The results are concluded in the Figure 1. Noted that constant domain and increasing domain interpretations make a significant difference.

We provide an informal guide to our latter results according to the expressivity of the bundled fragments. If a fragment can express, modulo some modal padding in a restricted way, both $\forall x \exists y \alpha$ and $\forall x \forall y \forall z \alpha$ in some form (like `EBBA` and `ABEBBE`), we can then prove that such a fragment is undecidable. If a fragment can express the essence of $\forall x \exists y \alpha$ but not $\forall x \forall y \forall z \alpha$ (like `EBBE` and `BE` over constant domain models) then we will prove that such fragments do not have finite model property. Finally, if a fragment cannot express the essence of $\forall x \exists y \alpha$ (like `ABEBBA` and `LBF`) then we will prove that it satisfies finite model property and give a tableau procedure.

References

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