Decidable Fragments of Term Modal Logic

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1 Introduction

Multi-modal First order modal logic (FOML) extends first order logic (FO) which modal operators of the form $\Box_i$ and $\Diamond_i$ where i comes from a finite set which is fixed in the syntax. This corresponds to the assumption that set of agents in the system is finite and fixed before hand. However, we encounter many settings in which the agent set is neither fixed nor static. For instance, in a server-client system we cannot bound the number of clients beforehand and keeps changing dynamically.
Motivated by these requirements, Fitting, Voronkov and Thalmann [FTV01] introduced Term Modal Logic (TML) which allows the modalities to be indexed by terms and these terms can be quantified. Thus we can assert properties of the flavor: *All eye-witnesses know who killed Mary* as $\exists y \forall x \big(Wit(x) \to \Box_x \; killed(y, Mary) \big)$.

Definition 1.1 *Let $\mathcal{P}$ be a collection of predicate symbols and let $\mathcal{V}$ be a countable set of variables. The syntax of TML is defined by:*

$$\phi ::= P(x_1, \ldots, x_n) \mid \neg\phi \mid (\phi \wedge \phi) \mid \exists x\ \phi \mid \Box_x\ \phi$$

where $P \in \mathcal{P}^n$ and $x, y, x_1, \ldots, x_n \in \mathcal{V}$.

Note that the only difference in the syntax of TML and FOML is in the index of the modality. In FOML, the index of the modal operators are fixed but in TML the modal indices are terms which are quantified over the underlying domain. Intuitively, this corresponds to assuming that the active domain at every world also forms the agent set at that world. Thus, the models of TML vary from that of FOML only in the labeling of the accessibility relation.

Definition 1.2 *A* TML *structure is a tuple* $\mathcal{M} = (\mathcal{W}, \mathcal{D}, \delta, \mathcal{R}, \rho)$ *where:* $\mathcal{W}$ *is the set of* worlds*;* $\mathcal{D}$ *is the set of* potential agent set*;* $\delta : W \to 2^D$ *maps every* $w \in \mathcal{W}$ *to a non-empty* local agent set*;* $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{D} \times \mathcal{W}$ *is the accessibility relation*[1] *and* $\rho : (\mathcal{W} \times \mathcal{P}) \mapsto \bigcup_n 2^{\mathcal{D}^n}$ *is the interpretation of predicates at every world.*

The semantics for TML is the same as FOML, except for the evaluation of $\Box_x \phi$ which is adapted to suit TML in the obvious way. Given a mapping for variables σ:

$$\mathcal{M}, w, \sigma \models \Box_x \phi \;\Leftrightarrow\; \text{for all } u \in \mathcal{W} \text{ if } (w, \sigma(x), u) \in \mathcal{R} \text{ then } \mathcal{M}, u, \sigma \models \phi$$

2 Decidable fragments

TML is very expressive in the sense that having just propositions as atoms makes the satisfiability problem undecidable [PR19]. Thus, finding decidable fragments of TML is as interesting and hard as the case of FOML.

2.1 Translation to FOML

From the syntax and semantics, it is clear that there is a close correspondence between FOML and TML. Indeed every TML formula can be translated into an FOML formula that preserves satisfiability [PR23]. Moreover such a translation preserves monodicity[2] [WZ01] and bundled fragments[3] [LPRW23]. Since these fragments of FOML are decidable, we can identify the corresponding decidable fragments of TML [PR23].

2.2 Two variable fragment

Note that the three variable fragment of TML is clearly undecidable (since three variable fragment FO is contained in TML). Further, since every one-variable formula is a monodic formula, the one variable fragment of TML is decidable. This leaves the two variable case (TML^2). In the absence of

[1] The monotonicity condition is imposed on the accessibility relation the same spirit as in FOML to handle the interpretation of free variables

[2] where there is at most one free variable in the scope of every modal operator

[3] where every quantifiers should be succeeded or preceded by a modal operator

constants or equality, the two variable fragment of TML is decidable [PR23]. This is surprising, since the two variable fragment of FOML (without constants or equality) is undecidable [WZ01].

From the perspective of FO though FO^2 is decidable, Grädel and Otto [GO99] show that the satisfiability problem for many of the natural extensions of FO^2 (like transitive closure, lfp) are undecidable[4]. In contrast, the 2-variable TML is yet another rare extension of FO^2 that remains decidable.

To prove the decidability, first we show that satisfiability of TML^2 can be reduced to the satisfiability PTML^2 (TML^2 restricted to only propositions as atoms) [PR23]. Hence it suffices to prove the decidablity of two variable propositional term modal logic (PTML^2). We then introduce a normal form for PTML^2 and prove bounded model property.

Normal form In [Fin75], Fine introduces a normal form for propositional modal logic (single agent) which is a disjunctive normal form (DNF) where every clause of the form $(\bigwedge\limits_i (s_i) \wedge \Box\alpha \wedge \bigwedge\limits_j \Diamond\beta_j)$ where every s_i is a propositional symbol or its negation and α, β_j are again in the normal form. For FO^2, we have Scott normal form [GKV97] where every FO^2 sentence has an equi-satisfiable sentence of the form $\forall x \forall y\ \phi \wedge \bigwedge\limits_i \forall x \exists y\ \psi_i$ where ϕ and every ψ_i are quantifier free.

For PTML^2, we introduce a combination of these two normal forms, which is a DNF formula where every clause is of the form:

$$\bigwedge_{i \leq a} s_i \wedge \bigwedge_{z \in \{x,y\}} (\Box_z \alpha \wedge \bigwedge_{j \leq m_z} \Diamond_z \beta_j) \ \wedge \bigwedge_{z \in \{x,y\}} (\forall z\ \gamma \wedge \bigwedge_{k \leq n_z} \exists z\ \delta_k) \ \wedge\ \forall x \forall y\ \phi \wedge \bigwedge_{l \leq b} \forall x \exists y\ \psi_l$$

where $a, m_x, m_y, n_x, n_y, b \geq 0$ and s_i denotes literals. Further, α and β_j are recursively in the normal form and the formulas $\gamma, \delta_k, \phi, \psi_l$ do not have quantifiers at the outermost level and all modal subformulas occurring in these formulas are (recursively) in the normal form.

Note that the first two conjuncts mimic the normal form for modal logic[Fin75] introduced by Fine and the last two conjuncts mimic the Scott normal form for FO^2 [GKV97]. The additional conjuncts handle the intermediate step where only one of the variable is quantified and the other is free. It can proved that every PTML^2 formula has a equi-satisfiable TML^2 in the normal form [PR23].

[4] the only decidable extension of FO^2 they consider is that of the counting quantifiers.

Bounded model property To show bounded model property for PTML^2, if a PTML^2 formula θ is satisfiable in a tree model, the strategy is to inductively come up with bounded agent models for every subtree of the given tree (based on types), starting from leaves to the root. While doing this, when we add new type based agents to a world at height h, to maintain monotonicity, we need to propagate the newly added agents throughout its descendants. Thus, the bounded agent property is proved using an argument that can be construed as *modal depth induction* over the 'classical' bounded model construction for FO^2.

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