Mezhirov's game for intuitionistic logic and its variations

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Game semantics allows us to look at basic logical concepts from another side. This approach to logic has a long history, there are plenty of different types of games: provability games, semantic games, etc [10,11]. And there is an interesting type of provability games called Mezhirov's game proposed by Iliya Mezhirov for intuitionistic logic of propositions (IPC) and Grzegorczyk modal logic (Grz) [1,2]. This idea was developed in many different directions; for example, in 2008 in the joint paper with N. Vereschagin a game semantics was given for affine and linear logic [3]. Independently G. Japaridze worked on game semantics for linear logic [4]. Mezhirov's games for minimal propositional logic (MPC), logic of functional frames ($KD!$) and logic of serial frames (KD) were introduced in 2021 by A. Pavlova [5].

Mezhirov's game semantics for intuitionistic logic is interesting because of its simplicity and strong connection with Kripke semantics and Kripke models. The game between Opponent and Proponent starts with a formula φ. And Proponent has a winning strategy iff φ is an intuitionistic tautology. The connection between the game and Kripke models manifests itself in building strategy for Opponent from a Kripke model (Opponent "walks" from one world of a model to another) and in the reconstruction of a model from Opponent's winning strategy (in which there exists a world where φ is false). And these procedures are connected to each other.

In my study, I try to generalize Mezhirov's result in two directions: to generalize to intuitionistic logic of predicates (introduce a game between Opponent and Proponent with at least the same connection with Kripke models or with special classes of them) and to the case of a connection not only between the game and tautologies of logic ($\vDash \varphi$), but also between the game and entailment from infinite sets of formulas ($T \vDash \varphi$).

The purpose of building such game was to get a theorem of kind "Proponent has a winning strategy in a special starting position (easily defined using an arbitrary set of formulas $\mathcal{O}_0$ and formula φ) iff $\mathcal{O}_0 \vDash \varphi$", where $\vDash$ is the semantic consequence defined by some class of predicate Kripke frames (φ is a semantic consequence of T in some class C of predicate Kripke frames iff for each Kripke model, based on a frame from the class C, if all formulas from T is true everywhere in this model, then φ is true in each world of the model). I initially thought about just logic of all Kripke models, i.e. it would be a game for intuitionistic logic of predicates directly. But it turned out that in such case some fundamental problems arise and it is natural to expand the logic (to use a smaller class of Kripke frames). Moreover, description of such variations (not just logic of all Kripke models) could be useful, since, in general, Kripke semantics for superintuitionistic predicate logic is rather weak (e.g. [9]). And I managed to get a description (based on the game I built) for several variations. So let me describe the rules of the game.

Let Ω be the elementary intuitionistic language (without function symbols; language will contain $\bot$, and the set of logical connectives will be $\{\to, \wedge, \vee\}$, where $\neg A$ will be considered as $A \to \bot$), and we will use Kripke models for intuitionistic logic of predicates [6,7] (I will call sets of constants in each world "individual domains" (or "the set of objects") and use symbol Δ). For the set of formulas Γ and set of objects (constants) Δ let $\mathcal{F}(\Gamma, \Delta) = \{P[c_1, ..., c_n] | P[x_1, ..., x_n]$ is a subformula of some formula from Γ and free variables of it are only $x_1, ..., x_n; c_i \in \Delta\}$ (so $\mathcal{F}$ in some ways is a set of all "subformulas" of formulas from Γ). Players Opponent and Proponent will be associated with their sets $\mathcal{O}$ and $\mathcal{P}$. The position in the game is a triple $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \Delta)$. In each position $\mathcal{C}$: $\mathcal{O}$ and $\mathcal{P}$ are subsets of $\mathcal{F}(\Gamma, \Delta)$, where Δ is taken from $\mathcal{C}$ (and can only expand in the game process) and Γ is fixed at the beginning of the game and does not change until the end and equals to $\mathcal{O}_0 \cup \{\varphi\}$ (where $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ is a starting position; Δ_0 is an exact set of all constants contained in formulas from Γ). Proponent moves by adding new formulas from $\mathcal{F}$ to $\mathcal{P}$, Opponent moves by expanding Δ (he can add nothing to Δ if he wants; and he can add to Δ not just constants from Ω) and than adding new formulas from $\mathcal{F}$ to $\mathcal{O}$.

The only thing left to define is who must move in a position $\mathcal{C}$. To do that, let us firstly define the notion of truth relation $\Vdash$ in $\mathcal{C}$ for formulas from $\mathcal{F}(\Gamma, \Delta)$:

$$\begin{array}{l} \mathcal{C} \nVdash \bot \\ \mathcal{C} \Vdash A[c_1, ..., c_n] \rightleftharpoons A[c_1, ..., c_n] \in \mathcal{O} \\ \mathcal{C} \Vdash \varphi \star \psi \rightleftharpoons \varphi \star \psi \in \mathcal{O} \cup \mathcal{P} \text{ and } (\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi),\ \star \in \{\rightarrow, \wedge, \vee\} \\ \mathcal{C} \Vdash qxP[x] \rightleftharpoons qxP[x] \in \mathcal{O} \cup \mathcal{P} \text{ and } q\alpha \in \Delta(\mathcal{C} \Vdash P[\alpha]),\ q \in \{\exists, \forall\} \end{array}$$

where A is a predicate symbol, $arity(A) = n$, $c_i \in \Delta$, P - formula with only one free variable. A star in the case of $(\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi)$ means logical meta connective and behaves like a classical connective (the same for q in $q\alpha \in \Delta$).

Let us call a formula from $\mathcal{P}$ Proponent's mistake if it is false in the current position (the same for $\mathcal{O}$ and Opponent). If Opponent has no mistakes but Proponent has, then Proponent moves. Otherwise, Opponent must move. And if after a turn of a fixed player he must move again, he loses. If the game goes on infinitely (each player manages to pass a turn to the other player each turn), Proponent wins; also let us call formulas from $\mathcal{O} \cup \mathcal{P}$ marked formulas.

Now let us consider several examples of the game. In the first game $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \varnothing)$, where $\varphi = \forall y \exists x (P[x] \to P[y])$. Because Δ is empty, there are no formulas in $\mathcal{F}$ of the form $\exists x(P[x] \to P[c])$, so Proponent has no mistakes, it's Opponent's turn. It is enough for him to just expand Δ, and it will be Proponent's turn. Proponent takes all formulas of the kind $\exists x(P[x] \to P[c])$ and $P[c] \to P[c]$ and passes turn to Opponent. He will do the same (expand Δ) and the game goes on infinitely.

In the second game $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \{c\})$, where $\varphi = \neg P[c] \to \neg\exists x P[x]$. φ is an implication, both sending and conclusion of it is not marked, therefore false in the current position. So φ is true, it's Opponent's turn. He expand Δ to $\{c, \alpha\}$ and add to $\mathcal{O}$ formulas $\neg P[c]$, $\exists x P[x]$, $P[\alpha]$. He might not add $\exists x P[x]$ to $\mathcal{O}$ and turn would still be passed to Proponent. But in this case Proponent would have an opportunity to add to $\mathcal{P}$ $\neg\exists x P[x]$ and make this formula true in position (because $\exists x P[x]$ would not be marked), and Opponent still would have needed to add $\exists x P[x]$. After that, Proponent will not be able to pass turn to Opponent, therefore, he will lose.

In the third game let $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \varnothing)$, where $\varphi = \forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]] \to \forall x P[x]$ (*Casari's schema or Casari's formula* [8]). Again φ is an implication, it's Opponent's turn. He needs to make sending false, so he expand Δ and add to $\mathcal{O}$ all formulas $(P[\alpha] \to \forall x P[x]) \to \forall x P[x]$ and sending of the φ: $\forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]]$. Than Propopent creates mistakes for Opponent by adding to $\mathcal{P}$ all formulas $(P[\alpha] \to \forall x P[x])$. To get rid of mistakes, Opponent needs to add all $P[\alpha]$, and than Proponent just add to $\mathcal{P}$ $\forall x P[x]$. The only thing Opponent can do now is to expand Δ and repeat everything again. As we can see, this is the winning strategy for Proponent, but φ is not true in all Kripke models. This formula will give us a useful class of Kripke frames (class of all Kripke frames in which Casari's formula is valid; let us call it Casari's class (Kripke frame is from Casari's class iff in every countable sequence of worlds ω_i their individual domains Δ_i remain finite and stabilize; so class of Casari's Kripke frames includes all Noetherian Kripke frames)).

It seems to me that, informally, this game (and Mezhirov's game for propositional intuitionistic logic) could be understood as follows: Opponent is trying to build a theory that belies Proponent's assertion that ϕ follows from $\mathcal{O}_0$ (or, in the case of $\mathcal{O}_0 = \varnothing$, is trying to build a theory that shows that Proponent's thesis (φ) is not valid in general). And this theory must be coherent (Opponent must have no mistakes), otherwise his approach is considered unsuccessful.

While getting closer to results, I should mention that there was some interest in considering variation of the game with only finite Δ (and Opponent can expand Δ adding only finite number of objects) because of better connection with Kripke models, so there appeared results for two variations of the game (described one (let us call it infinite) and the same but with finite Δ (finite variation)).

Theorem 1. *In the infinite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with possibly infinite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Noetherian Kripke frames.*

Theorem 2. *In the infinite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Casari's Kripke frames.*

These theorems lead, inter alia, to the fact that logics of Noetherian Kripke frames and of Casari's Kripke frames have the same weak entailment (entailment from finite sets of formulas). Similar results we can see for the finite variation.

Theorem 3. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with possibly infinite* $\mathcal{O}_0$*, but with only finite* Δ_0*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Noetherian Kripke frames with only finite individual domains* Δ *in each world.*

Theorem 4. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Casari's Kripke frames with only finite individual domains* Δ *in each world.*

Theorem 5. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all finite Kripke frames with only finite individual domains* Δ *in each world.*

As we can see, in the case of only finite individual domains in each world, logics of Noetherian Kripke frames, of Casari's Kripke frames and of finite Kripke frames have the same weak entailment.

As I mentioned, the main goal of this study was to find a game with strong connection with Kripke models. Partially, this has been achieved (proofs for all 5 theorems contain building a strategy for Opponent by "walking" from one world of a model to another); in addition, some connections have been established between weak entailment of logics of some classes. But the next step would be to find a triple: a class of Kripke frames, a game semantics and a calculus (probably, an infinitary sequent calculus) with the same strong entailment (entailment from not only finite, but from any sets of formulas). In this case, it is better to take a simpler class of Kripke frames in terms of the possible calculus for this class. So, because of this, Casari's class looks better than the Noetherian class. Therefore, I am trying right now to change rules of the game to get the same strong entailment as in logic of all Kripke frames from Casari's class.

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