# Kripke (in)completeness of predicate modal logics with axioms of bounded alternativity

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## 1 Introduction and preliminaries

The Kripke frame semantics is a valuable tool for the analysis of propositional modal logics. Despite by now well known phenomenon of propositional Kripke incompleteness, examples of Kripke incomplete propositional modal logics are rather contrived. By contrast, in predicate modal logic, Kripke incompleteness is common, and the precise boundaries of the usefulness of Kripke frame semantics are not as well understood as for propositional logics. At least in part, this situation arises since canonical models for predicate modal logics are not as well-behaved as canonical models for propositional logics. Even in cases when Kripke completeness can be obtained, a suitable Kripke frame is not canonical (in other words, canonicity is not as common in predicate modal logic as it is in propositional modal logic).

Here, we investigate both Kripke completeness and Kripke incompleteness in the context of logics $\mathbf{QAlt}_n$, which are minimal modal predicate logics containing the propositional axiom of bounded alternativity (here, $n \geqslant 1$),

$$\boldsymbol{alt}_n \quad = \quad \neg \bigwedge_{0 \leqslant i \leqslant n} \Diamond (p_i \wedge \bigwedge_{j \neq i} \neg p_j),$$

corresponding to the Kripke frame condition $|R(w)| \leqslant n$ whenever $w \in W$ ('$n$-alternativity'), as well as their minimal extensions containing axioms $T$ ('reflexivity') and 4 ('transitivity'). We show, using selective submodels [2], that both $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are strongly Kripke complete and, using Kripke bundle semantics [1, Chapter 5], that logics $\mathbf{QK4Alt}_n$ and $\mathbf{QS4Alt}_n$ are Kripke incomplete.[1]

We work with the language containing a countable supply of predicate letters of every arity, Boolean connectives, quantifier symbols, and a unary modal operator $\Box$. The definition of a formula is standard. We also use the abbreviation $\Box^{\leqslant n} A := \bigwedge_{i=0}^{n} \Box^i A$. By a predicate modal logic we mean a set of formulas including the classical predicate logic $\mathbf{QCL}$, the minimal propositional modal logic $\mathbf{K}$, and closed under Substitution, Modus Ponens, Generalization, and Necessitation. If $\Lambda$ is a propositional modal logic, the minimal predicate modal logic including $\Lambda$ is denoted by $\mathbf{Q}\Lambda$.

We briefly recall the Kripke frame semantics for predicate logics. A *Kripke frame* is a pair $(W, R)$ where $W \neq \varnothing$ and $R \subseteq W \times W$. A *predicate Kripke frame* is a tuple $\boldsymbol{F} = (F, D)$ where $F = (W, R)$ is a Kripke frame and $D = \{D_u \mid u \in W\}$ is a system of non-empty domains satisfying the condition that $D_u \subseteq D_v$ whenever $uRv$ ('expanding domains'). A model over a predicate Kripke frame $\boldsymbol{F}$ is a pair $M = (\boldsymbol{F}, \xi)$, where $\xi$ is a family $(\xi_u)_{u \in W}$ of maps such that $\xi_u(P^n) \subseteq D_u^n$, for each $n$-ary predicate letter $P^n$. The truth relation between models $M$, worlds $u$, and $D_u$-sentences $A$ (a $D_u$-sentence is obtained from a formula by substituting elements of $D_u$ for parameters of the formula) is standard; in particular,

- $M, u \models P(a_1, \ldots, a_n)$ if $(a_1, \ldots, a_n) \in \xi_u(P)$;
- $M, u \models \Box A(a_1, \ldots, a_n)$ if $M, v \models A(a_1, \ldots, a_n)$ whenever $v \in R(u)$.

A formula is true in a model if its universal closure is true at every world of the model. A formula is valid on a predicate Kripke frame if it is true in every model over the predicate frame.

[1] Strong completeness of logics $\mathbf{QAlt}_n$ was claimed, without proof, in [3]; here, we give a detailed proof.

If $\mathscr{C}$ is a class of predicate Kripke frames, the set of formulas valid on $\mathscr{C}$ is a modal predicate logic, denoted by $\mathtt{L}(\mathscr{C})$. If there exists a class $\mathscr{C}$ of predicate Kripke frames such that $L = \mathtt{L}(\mathscr{C})$, the logic $L$ is *Kripke complete*; if, in addition, every set of $L$-consistent formulas is satisfiable in a model over a predicate Kripke frame validating $L$, then $L$ is *strongly Kripke complete*.

## 2 Kripke completeness of $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$

Throughout this section, unless stated otherwise, $L$ is a predicate modal logic. For completeness proofs, we use languages extended with a set of constants of arbitrary cardinality. We assume, for now, a fixed universal set $\mathcal{S}$ of constants of infinite cardinality $\kappa$. A set $C \subseteq \mathcal{S}$ of constants is *$\mathcal{S}$-small* if $|\mathcal{S} - C| = \kappa$. If $C$ is a set of constants, a $C$-sentence is a sentence possibly containing constants from $C$. The set of all $C$-sentences is denoted by $\mathcal{L}(C)$. A *theory* is a set of $C$-sentences, for some $C \subseteq \mathcal{S}$. If $\Gamma$ is a theory, the set of constants occurring in $\Gamma$ is denoted by $C_\Gamma$; the set of all $C_\Gamma$-sentences is denoted by $\mathcal{L}(\Gamma)$.

A theory $\Gamma$ is *Henkin* if, for every sentence $\exists x\, A(x) \in \mathcal{L}(\Gamma)$, there exists $c \in C_\Gamma$ such that $\exists x A(x) \to A(c) \in \Gamma$. A maximal $L$-consistent theory is called *$L$-complete*. It can be easily checked that every $L$-complete Henkin theory $\Gamma$ has the *existence property*: $\exists x A(x) \in \Gamma \iff (\exists\, c \in C_\Gamma)\, A(c) \in \Gamma$.

Let $L$ be a first-order modal logic. An *$(L, \mathcal{S})$-place* (simply *$L$-place* if $\mathcal{S}$ is clear from the context or immaterial) is an $L$-complete Henkin theory with an $\mathcal{S}$-small set of constants.

**Lemma 2.1** *Every $L$-consistent theory with an $\mathcal{S}$-small set of constants is included into some $(L, \mathcal{S})$-place.*

The *canonical predicate Kripke frame* for $L$ w.r.t. $\mathcal{S}$ is the tuple $F_L^{\mathcal{S}} := (W_L^{\mathcal{S}}, R_L^{\mathcal{S}}, D_L^{\mathcal{S}})$, where $W_L^{\mathcal{S}}$ is the set of all $(L, \mathcal{S})$-places; $R_L^{\mathcal{S}}$ is the canonical accessibility relation on $W_L^{\mathcal{S}}$ defined as follows: $\Gamma\, R_L^{\mathcal{S}}\, \Delta$ if $\Box^{-}\Gamma \subseteq \Delta$; and $D_L^{\mathcal{S}} \colon W_L^{\mathcal{S}} \to 2^{\mathcal{S}}$ is the map defined by $D_L^{\mathcal{S}}(\Gamma) = C_\Gamma$. The *canonical Kripke model* for $L$ w.r.t. $\mathcal{S}$ is the tuple $M_L^{\mathcal{S}} := (F_L^{\mathcal{S}}, \xi_L^{\mathcal{S}})$, where $F_L^{\mathcal{S}}$ is the canonical predicate Kripke frame and $\xi_L^{\mathcal{S}}$ is the canonical valuation defined by$(\xi_L^{\mathcal{S}})_\Gamma(P_k^m) := \{\mathbf{c} \in C_\Gamma^m \mid P_k^m(\mathbf{c}) \in \Gamma\}$.

**Theorem 2.2** *For every $\Gamma \in W_L^{\mathcal{S}}$ and $A \in \mathcal{L}(C_\Gamma)$,*

$$\mathbf{M}_L^{\mathcal{S}}, \Gamma \models A \iff A \in \Gamma.$$

From now on the universal set of constants $\mathcal{S}$ is no longer fixed; from now on, it is a parameter. A logic $L$ is *canonical* if $F_L^{\mathcal{S}} \models L$, for every universal set $\mathcal{S}$ of constants. As in propositional logic, every canonical logic is strongly Kripke complete, but the examples of predicate canonical logics are scarce (see [1, Section 6.1]). In particular, it can be shown that logics $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are not canonical (proof idea: every world $\Gamma$ containing $\Diamond\top$ in canonical models for these logics sees infinitely many words containing constants outside of $C_\Gamma$). Nevertheless, these logics, as we next show, are Kripke complete. To prove this, we use the method of selective submodels [2, Section 6] resembling selective filtration in propositional modal logic and Tarski-Vaught test in classical model theory.

A Kripke model $M' = (W', R', D', \xi')$ is a *weak submodel* of a Kripke model $M = (W, R, D, \xi)$ if $W' \subseteq W$, $R' \subseteq R$, and, for every $w \in W'$, both $D_w = D'_w$ and $\xi'_w = \xi_w$. If, additionally, $M, w \models \Diamond A \Longrightarrow \exists u \in R'(w)\, M, u \models A$, for every $w \in W'$ and every $D_w$-sentence $A$, then $M'$ is a *selective* weak submodel of $M$.

**Lemma 2.3** *Let $M' = (W', R', D', \xi')$ be a selective weak submodel of $M = (W, R, D, \xi)$. Then, $M, w \models A \iff M', w \models A$, for every $w \in W'$ and every $D_w$-sentence $A$.*

A *quasi-canonical model* for a logic $L$ is a selective weak submodel of $M_L^{\mathcal{S}}$ (for some $\mathcal{S}$). A logic $L$ is *quasi-canonical* if, for every $L$-place $\Gamma$, there exists a quasi-canonical model over a predicate frame containing $\Gamma$ and validating $L$. By Theorem 2.2 and Lemma 2.3, if $M' = (W', R', D', \xi')$ a quasi-canonical model for $L$, then, $M', \Gamma \models A \iff A \in \Gamma$, for every $\Gamma \in W'$. Hence, due to Lemma 2.1,

**Theorem 2.4** *Every quasi-canonical predicate modal logic is strongly Kripke complete.*

**Theorem 2.5** *Let $L = \mathbf{QAlt}_n$ or $L = \mathbf{QTAlt}_n$, for some $n \geqslant 1$. Then, $L$ is quasi-canonical and, hence, strongly Kripke complete.*

**Proof.** Let $M_L = (W_L, R_L, D_L, \xi_L)$ be a canonical model for $L$, and let $\Gamma_0 \in W_L$. We obtain a selective submodel $M$ of $M_L$ over a frame validating $L$ and containing $\Gamma_0$. First, we prove the following:

**Lemma 2.6** *Let $\Gamma \in W_L$ and $X^\Gamma := \{\Delta \mid \Delta$ is $L$-complete & $\mathcal{L}(\Delta) = \mathcal{L}(\Gamma)$ & $\Box^-\Gamma \subseteq \Delta\}$. Then, $|X^\Gamma| \leqslant n$.*

**Proof.** Suppose that $\Delta_0, \ldots, \Delta_n$ are distinct theories from $X^\Gamma$. Since these theories are $L$-complete and $\mathcal{L}(\Delta_0) = \ldots = \mathcal{L}(\Delta_n) = \mathcal{L}(\Gamma)$, for each $i, j \in \{0, \ldots, n\}$ with $i \neq j$, there exists $A_{ij} \in \mathcal{L}(\Gamma)$ such that $A_{ij} \in \Delta_i$, but $A_{ij} \notin \Delta_j$. For every $i \in \{0, \ldots, n\}$, let $B_i = \bigwedge\limits_{j \neq i} (A_{ij} \wedge \neg A_{ji})$. Then, $B_i \in \Delta_j$ iff $i = j$. Hence, $\bigwedge\limits_{0 \leqslant i \leqslant n} \Diamond(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j) \in \Gamma$. But $\vdash_{\mathbf{QAlt}_n} \neg(\bigwedge\limits_{0 \leqslant i \leqslant n} \Diamond(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j))$. Thus, $\Gamma$ is $L$-inconsistent, contrary to the assumption. $\Box$

We now proceed with the proof of the theorem, distinguishing two cases.

Case $L = \mathbf{QAlt}_n$: We define the set $W$ of worlds and the accessibility relation $R$ of the model $M$ by recursion. Set $W_0 = \varnothing$, $W_1 = \{\Gamma_0\}$, and $R_0 = R_1 = \varnothing$. Suppose the sets $W_0, \ldots, W_k$ and the relations $R_0, \ldots, R_k$, for some $k < \omega$, have been defined. To define $W_{k+1}$ and $R_{k+1}$, consider, for each $\Gamma \in W_k - W_{k-1}$, the set $X^\Gamma$ defined in Lemma 2.6. By Lemma 2.6, $|X^\Gamma| \leqslant n$. By Lemma 2.1, for each $\Delta \in X^\Gamma$, there exists $\Delta' \in W_L$ such that $\Delta \subseteq \Delta'$; let $Y^\Gamma$ be the set containing exactly one such $\Delta' \in W_L$ for each $\Delta \in X^\Gamma$. Then, $|Y^\Gamma| \leqslant n$. By Existence Lemma and Lindenbaum lemma, for every sentence $A$, if $\Diamond A \in \Gamma$, then $(\exists \Delta_0 \in X^\Gamma)\, A \in \Delta$. Hence,

$$\Diamond A \in \Gamma \Longrightarrow (\exists \Delta \in Y^\Gamma)\, A \in \Delta. \tag{1}$$

Set $W_{k+1} = W_k \cup \bigcup\limits_{\Gamma \in W_k - W_{k-1}} Y^\Gamma$ and $R_{k+1} = R_k \cup \bigcup\limits_{\Gamma \in W_k - W_{k-1}} \big(\{\Gamma\} \times Y^\Gamma\big)$. As we have seen, if $\Gamma \in W_k - W_{k-1}$, then $|Y^\Gamma| \leqslant n$, and so $|R_{k+1}(\Gamma)| \leqslant n$. Observe that $R_{k+1} \subset R_L$. Lastly, let $W = \bigcup\limits_{k<\omega} W_k$ and $R = \bigcup\limits_{k<\omega} R_k$. Then, by (1),

$$\forall \Gamma \in W\, \forall A \in \mathcal{L}(\Gamma)\, \big(\Diamond A \in \Gamma \Longrightarrow (\exists \Delta \in R(\Gamma))\, A \in \Delta\big). \tag{2}$$

By definition of $R$ and Lemma 2.6, $|R(\Gamma)| \leqslant n$, for each $\Gamma \in W$. Also, $R \subseteq R_L$. Hence, $(W, R) \models \boldsymbol{alt}_n$. Lastly, let $M := M_L \restriction W$. Then, $(W, R, D) \models L$. Thus, $M$ is a submodel of $M_L$ over an $L$-frame containing $\Gamma_0$. By (2) and Theorem 2.2, $M$ is a selective submodel of $M_L$.

Case $L = \mathbf{QTAlt}_n$: The set $W$ and the relation $R$ are again defined by recursion. We set $W_0 = \{\Gamma_0\}$, $R_0 = R_1 = \{(\Gamma_0, \Gamma_0)\}$. We need to make sure that every relation $R_k$, and hence their union $R$, is reflexive. Suppose $R_k$ is reflexive, for some $k < \omega$. Since $R_L$ is reflexive, it follows that $\Gamma \in X_\Gamma$. We pick the $L$-complete set $\Gamma' \in Y^\Gamma$ so that $\Gamma' = \Gamma$. Then, $R_{k+1}$ is reflexive. Hence, $R$ is reflexive, and so and $(W, R, D) \models L$. $\Box$

# 3 Kripke incompleteness of $\mathbf{QK4Alt}_n$ and $\mathbf{QS4Alt}_n$

To prove Kripke incompleteness of logics $\mathbf{QK4Alt}_n$ and $\mathbf{QS4Alt}_n$, we use the semantics of Kripke bundles [1, Chapter 5]. A *Kripke bundle* is a tuple $\mathbb{F} = (F, D, \rho)$, where $F = (W, R)$ is a Kripke frame, $D = \{D_u \mid u \in W\}$ is a family of non-empty disjoint domains, and $\rho = \{\rho_{uv} \mid (u, v) \in R\}$ is a family of inheritance relations $\rho_{uv} \subseteq D_u \times D_v$ satisfying the constraint that $\rho_{uv}(a) \neq \varnothing$ whenever $uRv$ and $a \in D_u$. Models over Kripke bundles are defined analogously to models over Kripke frames. The truth clause for formulas beginning with $\Box$ is as follows: $M, u \models \Box A(a_1, \ldots, a_n)$, with distinct $a_1, \ldots, a_n \in D_u$, if

$$\forall v \in R(u) \forall b_1 \in \rho_{uv}(a_1) \ldots \forall b_n \in \rho_{uv}(a_n)\, M, v \models A(b_1, \ldots, b_n).$$

A formula is *true* in Kripke bundle model if its universal closure is true at every world of the model. A formula $A$ is *strongly valid* in a Kripke bundle $\mathbb{F}$ (notation: $\mathbb{F} \Vdash A$) if every substitution instance of $A$ is true in every model over $\mathbb{F}$. The following is well known [1, Proposition 5.2.12]:

**Proposition 3.1** *Let $\mathbb{F}$ be a Kripke bundle. Then the set $\{A \mid \mathbb{F} \Vdash A\}$ is a modal predicate logic.*

With every Kripke bundle $\mathbb{F} = (W, R, D, \rho)$, we associate a family $\{(W_n, R_n) \mid n < \omega\}$ of Kripke frames: put $D_0 := W$ and $R_0 := R$; put $D_1 := \bigcup\{D_u \mid u \in W\}$ and $R_1 := \bigcup\{\rho_{uv} \mid uRv\}$; for every $n > 1$, put $D_n := \bigcup\{D_u^n \mid u \in W\}$ and

$$R_n \quad := \quad \{(\mathbf{a}, \mathbf{c}) \in D_n \times D_n \mid \forall j \, a_j R_1 b_j \text{ and } \forall j, k \, (a_j = a_k \Rightarrow b_j = b_k)\}.$$

The following is well known [1, Proposition 5.3.7]:

**Proposition 3.2** *Let $\mathbb{F}$ be a Kripke bundle and $A$ a modal propositional formula. Then, $\mathbb{F} \Vdash A$ iff $F_n \models A$, for every $n < \omega$.*

**Theorem 3.3** *Let $L = \mathbf{QK4Alt}_n$ or $L = \mathbf{QS4Alt}_n$, for some $n \geqslant 1$. Then, $L$ is Kripke incomplete.*

To prove incompleteness of $\mathbf{QK4Alt}_n$, we make use of the formula $\forall ref := \forall x\,(\Box P(x) \to P(x))$. We show that every Kripke predicate frame validating $\mathbf{QK4Alt}_n$ validates $A_n := \Diamond^{\leqslant n+1}\top \to \Diamond\forall ref$, but $A_n \notin \mathbf{QK4Alt}_n$.

Suppose that $\boldsymbol{F} = (W, R, D) \models \mathbf{QK4Alt}_n$, and so $R$ is transitive and $n$-alternative. Let $M$ be a model over $\boldsymbol{F}$ and $u_0 \in W$. Assume that $M, u_0 \models \Diamond^{\leqslant n+1}$. Then, there exist $u_1, \ldots, u_{n+1} \in W$ such that $u_0 R u_1 R \ldots R u_{n+1}$. Since $R$ is $n$-alternative, there exist $k, j \leqslant n+1$ such that $k \neq j$ and $u_k = u_j$. But then $u_k$ is reflexive, and so $M, u_k \models \forall ref$. Hence, $M, u_0 \models \Diamond\forall ref$ and so $M, u_0 \models A_n$.

To show that $A \notin \mathbf{QK4Alt}_n$, in view of Proposition 3.1, it suffices to obtain a Kripke bundle strongly validating $\mathbf{QK4Alt}_n$, but refuting $A$. Define $W = \{u\}$, $R = \{(u, u)\}$, $D_u = \{a, b\}$, and $\rho = \{(a, b), (b, b)\}$. Put $\mathbb{F}_0 = (W, R, D, \rho)$. It should be clear that $\mathbb{F}_0$ is a Kripke bundle. To see that $\mathbb{F}_0 \not\Vdash A$, consider the model $M_0 = (\mathbb{F}_0, \xi)$ with $\xi_u(P) = \{b\}$. Since $M_0, u \models P(b)$, the world $u$ is reflexive, and $b$ is the unique inheritor of $a$, it follows that $M_0, u \models \Box P(a)$. Since $M_0, u \not\models P(a)$, it follows that $M_0, u \not\models \Box P(a) \to P(a)$ and so $M_0, u \not\models \Diamond\forall ref$. On the other hand, since $R$ is serial, $M_0, u \models \Diamond^{\leqslant n+1}\top$. Hence, $\mathbb{F}_0 \not\models A_n$.

It remains to prove that $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_n$. We use Proposition 3.2 to prove that $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_1$ and hence $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_n$, for every $n \geqslant 1$. It should be clear that $F_0 = (W, R) \models \mathbf{K4Alt}_1$. Let $n \geqslant 1$ and $\mathbf{d}, \mathbf{e} \in D_n$. Then, $\mathbf{d} R_n \mathbf{e}$ iff $\forall j \, e_j = b$; hence, every $\mathbf{d} \in D_n$ has exactly one $R_n$-successor, $\mathbf{b}$, and so $R_n$ is transitive and 1-alternative (in fact, functional). Thus, $F_n \models \mathbf{K4Alt}_1$, for every $n < \omega$. Hence, by Proposition 3.2, $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_1$ and so $\mathbb{F}_0 \Vdash \mathbf{QK4Alt}_n$, for every $n \geqslant 1$.

The proof for $\mathbf{QS4Alt}_n$ is analogous. Instead of the formula $A_n$, we use $\Diamond\Box\forall x\,(\Diamond\Box P(x) \to P(x))$, and instead of the Kripke bundle $\mathbb{F}_0$, we use the Kripke bundle $\mathbb{F}_1$ defined as follows: $W = \{u\}$, $R = \{(u, u)\}$, $D_u = \{a, b\}$, $\rho = \{(a, a), (a, b), (b, b)\}$, and $\mathbb{F}_1 = (W, R, D, \rho)$.

# References


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