ON ALGORITHMIC EXPRESSIVITY OF FINITE-VARIABLE FRAGMENTS OF INTUITIONISTIC MODAL LOGICS

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1. Introduction

Modal and intuitionistic propositional logics are often poly-time embeddable into their own fragments with a few variables (typically, zero, one, or two), and similar embeddings are sometimes constructed of fragments of logics with special properties into finite-variable fragments of those logics. The literature on the topic is quite extensive [2, 27, 11, 12, 4, 29] and includes contributions by the authors of this paper [3, 14, 15, 16, 17, 18, 20, 19, 21, 22, 23, 24, 25].

As a result, the validity problem for such fragments is as computationally hard as the validity problem for the full logic. (In general, modal and superintuitionistic propositional logics, even linearly approximable ones, may have arbitrarily hard fragments with a few variables since, for every set $A \subseteq \mathbb{N}$, one can construct [26] a linearly approximable logic whose fragment with a few variables (typically zero, one, or two) recursively encodes A. We obtain here similar embeddings for the intuitionistic modal logics **FS** and **MIPC**, introduced by, respectively, Fisher Servi [7] and Prior [13]. These logics have been introduced as counterparts of bimodal propositional logics, and can also be viewed as fragments of the predicate intuitionistic logic **QInt** (for details, see [10]); we note that this is not the only approach to constructing modal intuitionistic logics, cf. [5, 6, 28]. The complexity of **FS** and **MIPC** remains unresolved, but the results presented here show that single-variable fragments of these logics have the same complexity as the full logics.

2. Preliminaries

The intuitionistic modal language contains a countable set $\mathcal{P}$ of propositional variables, the constant $\bot$, binary connectives $\circ$, $\wedge$, and $\to$, and unary modal connectives $\Diamond$ and $\Box$. Formulas are defined in the usual way. A formula is *positive* if it does not contain occurrences of $\bot$. The set of propositional variables of a formula φ is denoted by $var\,\varphi$. The result of substituting a formula ψ for a variable p into a formula φ is denoted by $[\psi/p]\varphi$. The modal depth of a formula φ, denoted by $md\,\varphi$, is the maximal number of nested modal connectives in φ. The length of a formula φ, defined as the number of symbols in φ (with the binary encoding of variables), is denoted by $|\varphi|$.

We define the logics **FS** and **MIPC** semantically. A *Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of *worlds* and R is a partial order on W. An **FS**-*frame* is a triple $\mathfrak{F} = \langle W, R, \delta\rangle$, where $\langle W, R\rangle$ is a Kripke frame and δ is a map associating with each $w \in W$ a structure $\langle \Delta_w, S_w\rangle$, with Δ_w being a non-empty set of *points* and S_w a binary relation on Δ_w such that, for every $w, v \in W$,

$$v \in R(w) \;\Rightarrow\; \Delta_w \subseteq \Delta_v \quad \text{and} \quad S_w \subseteq S_v$$

An **FS**-frame $\mathfrak{F} = \langle W, R, \delta\rangle$ is an **MIPC**-*frame* if $S_w = \Delta_w \times \Delta_w$, for every $w \in W$. A *valuation* on an **FS**-frame $\langle W, R, \delta\rangle$ is a map associating with each $w \in W$ and each $p \in \mathcal{P}$ a subset $V(w, p)$ of Δ_w in such a way that

$$v \in R(w) \;\Rightarrow\; V(w, p) \subseteq V(v, p).$$

The pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where $\mathfrak{F}$ is an **FS**-frame and V a valuation on $\mathfrak{F}$, is called an **FS**-*model*. An **MIPC**-*model* is an **FS**-model over an **MIPC**-frame. The *truth-relation* $\models$ is defined by recursion (here, $\mathfrak{M}$ is a model, $w \in W$, $x \in \Delta_w$, and φ is a formula):

- $\mathfrak{M}, w, x \models p \quad\leftrightharpoons\quad x \in V(w, p) \quad$ if $p \in \mathcal{P}$;
- $\mathfrak{M}, w, x \not\models \bot$;
- $\mathfrak{M}, w, x \models \varphi_1 \circ \varphi_2 \quad\leftrightharpoons\quad \mathfrak{M}, w, x \models \varphi_1$ and $\mathfrak{M}, w, x \models \varphi_2$;
- $\mathfrak{M}, w, x \models \varphi_1 \wedge \varphi_2 \quad\leftrightharpoons\quad \mathfrak{M}, w, x \models \varphi_1$ or $\mathfrak{M}, w, x \models \varphi_2$;
- $\mathfrak{M}, w, x \models \varphi_1 \to \varphi_2 \quad\leftrightharpoons\quad \mathfrak{M}, v, x \not\models \varphi_1$ or $\mathfrak{M}, v, x \models \varphi_2$ whenever $v \in R(w)$;
- $\mathfrak{M}, w, x \models \Diamond\varphi_1 \quad\leftrightharpoons\quad \mathfrak{M}, w, y \models \varphi_1$, for some $y \in S_w(x)$;
- $\mathfrak{M}, w, x \models \Box\varphi_1 \quad\leftrightharpoons\quad \mathfrak{M}, v, y \models \varphi_1$ whenever $v \in R(w)$ and $y \in S_v(x)$.

A formula φ is *true* in a model $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) if $\mathfrak{M}, w, x \models \varphi$, for every world w of $\mathfrak{M}$ and every point x of w. A formula φ is *valid* an **FS**-frame $\mathfrak{F}$ if φ is true in every model over $\mathfrak{F}$. Logics **FS** and **MIPC** are defined as sets of formulas valid on, respectively, every **FS**-frame and every **MIPC**-frame.

3. Main results

In this section, we prove that logics **FS** and **MIPC** are polynomial-time embeddable into their own fragments with a single propositional variable. We first poly-time embed these logics into their own positive fragments. Let φ be a formula and $f \in \mathcal{P} \setminus var\,\varphi$. Define

$$\varphi^f = [f/\bot]\varphi; \qquad F_1 = \Diamond^{\leqslant md\,\varphi} f \to f; \qquad F_2 = f \to \Box^{\leqslant md\,\varphi} f; \qquad F_3 = \bigwedge_{p \in var\,\varphi} \Box^{\leqslant md\,\varphi}(f \to p),$$

and put $F = F_1 \circ F_2 \circ F_3$.

Lemma 1. *Let φ be a formula, $f \in \mathcal{P} \setminus var\,\varphi$, and $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$. Then,*

$$\varphi \in L \iff F \to \varphi^f \in L.$$

Since φ^f and F are both positive, the map $e\colon \varphi \mapsto (F \to \varphi^f)$ embeds **FS** and **MIPC** into their own positive fragments.

We next define a polytime computable function $\cdot^*$ from the set of positive formulas to the set of one-variable positive formulas and show that, for $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$ and every positive φ,

$$\varphi^* \in L \iff \varphi \in L.$$

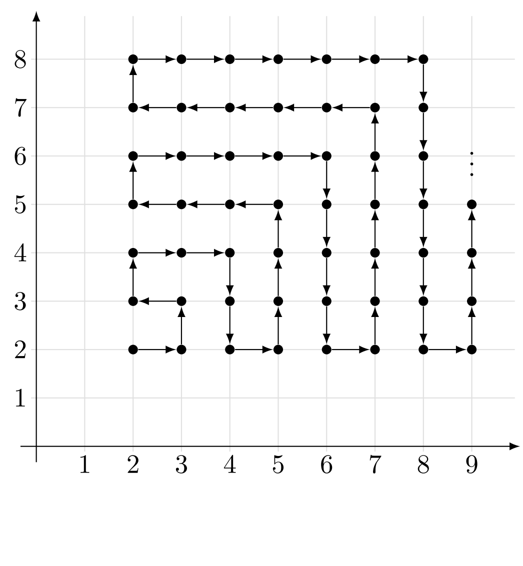
Hence, for every φ,

$$\varphi \in L \iff e(\varphi) \in L \iff e(\varphi)^* \in L.$$

The formula φ^* shall be obtain from φ using a substitution. We next define the formulas that shall be substituted for propositional variables of φ. These formulas, except G_1, G_2, and G_3, are divided into 'levels', indexed by elements of $\mathbb{N}$; formulas of level 0 are denoted A_i^0 or B_i^0, those of level 1, by A_i^1 and B_i^1, etc. We begin with G_1, G_2, and G_3, as well as formulas of levels 0 and 1:

$$\begin{array}{lcllcl}
G_1 & = & \Diamond p; & A_1^1 & = & A_1^0 \circ A_2^0 \to B_1^0 \wedge B_2^0; \\
G_2 & = & \Diamond p \to p; & A_2^1 & = & A_1^0 \circ B_1^0 \to A_2^0 \wedge B_2^0; \\
G_3 & = & p \to \Box p; & A_3^1 & = & A_1^0 \circ B_2^0 \to A_2^0 \wedge B_1^0; \\
A_1^0 & = & G_2 \to G_1 \wedge G_3; & B_1^1 & = & A_2^0 \circ B_1^0 \to A_1^0 \wedge B_2^0; \\
A_2^0 & = & G_3 \to G_1 \wedge G_2; & B_2^1 & = & A_2^0 \circ B_2^0 \to A_1^0 \wedge B_1^0; \\
B_1^0 & = & G_1 \to G_2 \wedge G_3; & B_3^1 & = & B_1^0 \circ B_2^0 \to A_1^0 \wedge A_2^0. \\
B_2^0 & = & A_1^0 \circ A_2^0 \circ B_1^0 \to G_1 \wedge G_2 \wedge G_3; & & &
\end{array}$$

We proceed by recursion. Let $k \geqslant 1$. Suppose the formulas $A_1^k, \ldots, A_{n_k}^k$ and $B_1^k, \ldots, B_{n_k}^k$ have been defined, with n_k being the number of formulas of the form A_i^k and, also, the number of formulas of the form B_i^k (e.g., if $k = 1$, then $n_k = 3$; the recursive definition for the cases where $k \geqslant 2$ is to be given). Define a linear order $\prec$ on the set $(\mathbb{N} \setminus \{0,1\}) \times (\mathbb{N} \setminus \{0,1\})$ as in the following picture, so that $\langle i, j \rangle \prec \langle i', j' \rangle$ if, and only if, there exists a path along one or more arrows from $\langle i, j \rangle$ to $\langle i', j' \rangle$:



We can then define an enumeration g of the pairs $\langle i,j\rangle \in (\mathbb{N}\setminus\{0,1\})\times(\mathbb{N}\setminus\{0,1\})$ according to $\prec$, i.e., so that $g(2,2)=1$, $g(3,2)=2$, $g(3,3)=3$, $g(2,3)=4$, etc. Now, for every $i,j\in\{2,\dots,n_k\}$, define

$$A^{k+1}_{g(i,j)} \;=\; A^k_1 \to B^k_1 \wedge A^k_i \wedge B^k_j; \qquad B^{k+1}_{g(i,j)} \;=\; B^k_1 \to A^k_1 \wedge A^k_i \wedge B^k_j,$$

and let n_{k+1} be the number of the formulas of the form A^{k+1}_i (which is the same as the number of formulas of the form B^{k+1}_i) so defined; notice that $n_{k+1}=(n_k-1)^2$. This completes the recursive definition of A^k_i and B^k_i.

Next, put

$$l_0 \;=\; |A^0_1|+|B^0_1|+|A^0_2|+|B^0_2|.$$

Lemma 2. *There exists $k_0\in\mathbb{N}$ such that $n_k > l_0\cdot 5^k$ whenever $k\geqslant k_0$.*

Now, let φ be a positive formula with $var\,\varphi=\{p_1,\dots,p_s\}$. Let k_φ be the least integer k such that $|\varphi| < l_0\cdot 5^k$. By Lemma 2, $n_{k_\varphi+k_0} > l_0\cdot 5^{k_\varphi+k_0}$; hence,

$$n_{k_\varphi+k_0} > l_0\cdot 5^{k_\varphi+k_0} > 5^{k_0}\cdot|\varphi| > |\varphi| \geqslant s.$$

Lastly, define φ^* to be the result of substituting into φ, for every $r\in\{1,\dots,s\}$, the formula $A^{k_\varphi+k_0}_r \wedge B^{k_\varphi+k_0}_r$ for the variable p_r (this substitution is well defined since $n_{k_\varphi+k_0} > s$).

We next show that φ^* is poly-time computable from φ.

Lemma 3. *For every $k\geqslant 0$ and every $i\in\{1,\dots,n_k\}$,*

$$|A^k_i| < l_0\cdot 5^k \quad\textit{and}\quad |B^k_i| < l_0\cdot 5^k.$$

Lemma 4. *The formula φ^* is computable in time polynomial in $|\varphi|$.*

Proof. It suffices to show that $|\varphi^*|$ is polynomial in $|\varphi|$. Since k_φ is the least integer k such that $|\varphi| < l_0\cdot 5^k$, surely $l_0\cdot 5^{k_\varphi-1}\leqslant|\varphi|$, and so

$$l_0\cdot 5^{k_\varphi+k_0} \leqslant 5^{k_0+1}|\varphi|.$$

By Lemma 3, for every $i\in\{1,\dots,n_{k_\varphi+k_0}\}$,

$$|A^{k_\varphi+k_0}_i| < l_0\cdot 5^{k_\varphi+k_0} \leqslant 5^{k_0+1}|\varphi| \quad\text{and}\quad |B^{k_\varphi+k_0}_i| < l_0\cdot 5^{k_\varphi+k_0} \leqslant 5^{k_0+1}|\varphi|.$$

Hence, $|\varphi^*| < 2\cdot 5^{k_0+1}|\varphi|^2$. □

To obtain the desired result, it remains to show the following:

Lemma 5. *Let $L\in\{\mathbf{FS},\mathbf{MIPC}\}$. Then, for every positive formula φ,*

$$\varphi\in L \quad\Longleftrightarrow\quad \varphi^*\in L.$$

From Lemmas 1, 4, and 5, we immediately obtain the following:

Theorem 6. *Let $L\in\{\mathbf{FS},\mathbf{MIPC}\}$. Then, there exists a polynomial-time computable function embedding L into its own positive one-variable fragment.*

Corollary 7. *Let $L\in\{\mathbf{FS},\mathbf{MIPC}\}$. Then, the positive one-variable fragment of L is polytime-equivalent to L.*

The results presented here are not immediately applicable to obtaining the computational complexity of finite-variable fragments of intuitionistic modal logics since the complexity of full logics remains unknown (we are only aware of decidability results [9, 31, 30, 1, 8] for modal intuitionistic logics).

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