ALGORITHMIC COMPLEXITY OF MONADIC MULTIMODAL PREDICATE LOGICS WITH EQUALITY OVER FINITE KRIPKE FRAMES

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1. Introduction

Monadic modal and superintuitionistic logics are, as a rule, undecidable in very poor vocabularies—in most cases, to prove undecidability, it suffices to use a single monadic predicate letter and two or three individual variables [9, 11, 12, 13, 14, 8] (for undecidability of related fragments of the classical logics, see [17, 10, 6, 7]). At the same time, the monadic fragment with equality of the classical predicate logic $\mathbf{QCl}^=$ is decidable [1]. Hence, it is of interest to identify settings where decidability can be obtained.

Proofs of undecidability of monadic fragments usually rely on the so-called the "Kripke trick" [4], a simulation of a subformula $P(x, y)$ of a classical formula with a monadic modal formula $\Diamond(Q_1(x)\wedge Q_2(y))$. Hence, indentifying decidable fragments involves discovering setting where the Kripke trick is not applicable. This has been done syntactically by Wolter and Zakharyaschev [18], who discovered monodic fragments (note that these differ from monadic fragments) disallowing the application of modalities to formulas with more than one parameter. Here, we consider a simple semantical setting where the Kripke trick does not work: the monadic predicate logic with equality of a Kripke frame with finitely many possible worlds (but, possibly, infinite domains). We also obtain precise complexity bounds for monadic logics of classes of Kripke frames with finitely many possible worlds. This is of interest since precise bounds beyond Σ^0_1 hardness are scarce in the literature on predicate modal logic. The observations presented here are generalizations to the multimodal settings of results from in [5, 15].

2. Preliminaries

We consider the n-modal, where $n \in \mathbb{N}^+$, predicate language $\mathcal{L}_n$ obtained by adding to the classical predicate language $\mathcal{L}_0$ unary modal connectives $\Box_1, \ldots, \Box_n$, as well as the language $\mathcal{L}_n^=$ obtained by adding to $\mathcal{L}_n$ a designated binary predicate letter $=$. The definitions of formulas are standard. A *monadic $\mathcal{L}_n$-formula* contains only monadic predicate letters. A *monadic $\mathcal{L}_n$-formula with equality* contains only monadic predicate letters and $=$.

By a normal n-modal predicate logic we mean a set of $\mathcal{L}_n$-formulas including the classical predicate logic $\mathbf{QCl}$ and the minimal normal n-modal propositional logic $\mathbf{K}_n$ and closed under Modus Ponens, Substitution, Necessitation, and Generalisation. A normal n-modal predicate logic with equality additionally contains the classical equality axioms. The minimal logic containing $\mathbf{QCl}$ and the n-modal propositional logic L is denoted by $\mathbf{Q}L$; the minimal extension of $\mathbf{Q}L$ containing the classical equality axioms is denoted by $\mathbf{Q}^=L$. The minimal extension of an n-modal predicate logic L containing, for each $k \in \{1, \ldots, n\}$, the Barcan formula $\boldsymbol{bf}_k = \forall x\, \Box_k P(x) \to \Box_k \forall x\, P(x)$, is denoted by $L.\mathbf{bf}$.

A *fusion* of 1-modal propositional logics $L_1, \ldots, L_n$ is the logic $L_1 * \ldots * L_n = \mathbf{K}_n \oplus (L_1 \cup L'_2 \cup \ldots \cup L'_n)$, where L'_i is obtained from L_i by replacing every occurrence of $\Box_1$ with $\Box_i$.

We use the framework of Kripke semantics for logics with and without equality (for more details, see [3]; our terminology differs from that adopted in [3]). There are two natural way to extend the well-known Kripke semantics for logics without equality to logics with equality; to treat equality as identity or as hereditary congruence. Unlike the classical logic, these two treatments of equality are not equivalent: the formula $x \neq y \to \Box_k(x \neq y)$ is valid if $=$ is interpreted as identity, but not valid if $=$ is interpreted as hereditary congruence. Here, except in Section 4, we treat equality as congruence.

A Kripke n-frame is a tuple $\mathfrak{F} = \langle W, R_1, \ldots, R_n\rangle$, where W is a non-empty set of worlds and $R_1, \ldots, R_n$ are binary accessibility relations on W. An augmented n-frame is a tuple $\boldsymbol{\mathfrak{F}} = \langle \mathfrak{F}, D\rangle$, where $\mathfrak{F}$ is a Kripke n-frame and D a family $(D_w)_{w\in W}$ of non-empty domains satisfying the expanding domains condition: for every $w, v \in W$,

$$(E) \quad wR_kv \implies D_w \subseteq D_v.$$

The condition (E) is required for soundness and completeness of predicate modal logics whose $\mathcal{L}_0$-fragment is **QCl**. If an augmented n-frame satisfies

$$(C) \quad wR_kv \quad \Longrightarrow \quad D_w = D_v,$$

then it is called a locally constant augmented n-frame. A model is a tuple $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$, where $\mathfrak{F}$ is an augmented n-frame and I is a family $(I_w)_{w\in W}$ of interpretations of predicate letters: $I_w(P) \subseteq D_w^m$, for every m-ary letter P.

An augmented n-frame with equality is a tuple $\mathfrak{F} = \langle \mathfrak{F}, D, \equiv\rangle$, where $\langle \mathfrak{F}, D\rangle$ is an augmented n-frame and $\equiv$ is a family $(\equiv_w)_{w\in W}$ of equivalence relations, with $\equiv_w \subseteq D_w^2$ whenever $w \in W$, satisfying the heredity condition: for every $w, v \in W$,

$$(H) \quad wR_kv \quad \Longrightarrow \quad \equiv_w \subseteq \equiv_v.$$

The condition (H) corresponds to the formula $x = y \to \Box_k(x = y)$, which belongs to $\mathbf{Q^{=}K}$, and hence to every normal modal predicate logic with equality. A model with equality is a tuple $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$, where $\mathfrak{F}$ is an augmented n-frame with equality and I is a family $(I_w)_{w\in W}$ of interpretations of predicate letters such that $\equiv_w$ is a congruence on the classical model $M_w = \langle D_w, I_w\rangle$.

The truth relation for $\mathcal{L}_n$ and $\mathcal{L}_n^{=}$ is defined by usual way; in particular, if $a, b \in D_w$ and $\bar{c}$ is a list of elements of D_w of a suitable length, then

$$\begin{array}{lcl} \mathfrak{M}, w \models a = b & \leftrightharpoons & a \equiv_w b; \\ \mathfrak{M}, w \models P(\bar{c}) & \leftrightharpoons & \bar{c} \in I_w(P); \\ \mathfrak{M}, w \models \forall x\, \varphi(x, \bar{c}) & \leftrightharpoons & \mathfrak{M}, w \models \varphi(d, \bar{c}), \text{ for every } d \in D_w; \\ \mathfrak{M}, w \models \Box_k \varphi(\bar{c}) & \leftrightharpoons & \mathfrak{M}, v \models \varphi(\bar{c}), \text{ for every } v \in R_k(w). \end{array}$$

The following definitions concern both $\mathcal{L}_n$ and $\mathcal{L}_n^{=}$; for the latter, all the models and augmented frames should be understood as those with equality. A formula φ is true at a world w if a universal closure of φ is true at w. A formula φ is true in a model $\mathfrak{M}$ if φ true at every world of $\mathfrak{M}$; φ is valid on an augmented n-frame $\mathfrak{F}$ if it is true in every model over $\mathfrak{F}$; φ is valid on a Kripke n-frame $\mathfrak{F}$ if φ is valid on every augmented n-frame over $\mathfrak{F}$; φ is valid on a class $\mathscr{C}$ of augmented frames if it is valid on every augmented frame from $\mathscr{C}$.

If $\mathscr{C}$ is a class of Kripke n-frames and $\mathfrak{F}$ is a Kripke frame, then

- $L(\mathscr{C})$ denotes the set of $\mathcal{L}_n$-formulas valid on $\mathscr{C}$;
- $L_c(\mathscr{C})$ denotes the set of $\mathcal{L}_n$-formulas valid on every locally constant augmented n-frame over a Kripke frame from $\mathscr{C}$;
- $L^{=}(\mathscr{C})$ denotes the set of $\mathcal{L}_n^{=}$-formulas valid on $\mathscr{C}$;
- $L_c^{=}(\mathscr{C})$ denotes the set of $\mathcal{L}_n^{=}$-formulas valid on every locally constant augmented n-frame with equality over a Kripke frame from $\mathscr{C}$.

We write $L^{=}(\mathfrak{F})$ and $L_c^{=}(\mathfrak{F})$ rather than $L^{=}(\{\mathfrak{F}\})$ and $L_c^{=}(\{\mathfrak{F}\})$, respectively.

A Kripke n-frame $\langle W, R_1, \ldots, R_n\rangle$ is finite if W is a finite set. If L is an n-modal predicate logic (with or without equality), then L^{wfin} denotes the set of formulas valid on every finite Kripke frame validating L; this set is a normal n-modal predicate logic.

3. Main results

The following is our main technical result:

Proposition 1. *Let $\mathfrak{F}$ be a finite Kripke frame. Then, the monadic fragments with equality of the logics $L^{=}(\mathfrak{F})$ and $L_c^{=}(\mathfrak{F})$ are both decidable.*

From Proposition 1 we obtain the following:

Theorem 2. *Let $\mathscr{C}$ be a recursively enumerable class of finite Kripke n-frames. Then the monadic fragments with equality of the logics $L^{=}(\mathscr{C})$ and $L_c^{=}(\mathscr{C})$ are both in Π_1^0.*

It is known [12, Theorem 3.9] that, if L is a logic from one of the intervals $[\mathbf{QK}^{wfin}, \mathbf{QGL.3.bf}^{wfin}]$, $[\mathbf{QK}^{wfin}, \mathbf{QGrz.3.bf}^{wfin}]$ or $[\mathbf{QK}^{wfin}, \mathbf{QS5}^{wfin}]$, then the monadic fragment of L is Π_1^0-hard. Hence, Theorem 2, together with the transfer of completeness theorem for fusions [2, Theorem 4.1], give us the following:

Corollary 3. *Let $L = L_1 * \ldots * L_n$, where $L_1, \ldots, L_n$ are normal 1-modal propositional logics, such that*

- *$L_i \subseteq$ **S5** or $L_i \subseteq$ **GL.3** or $L_i \subseteq$ **Grz.3**, for some $i \in \{1, \ldots, n\}$;*
- *the class of finite Kripke frames validating L is recursively enumerable.*

Then, the monadic fragments of the logics $\mathbf{Q}L^{wfin}$, $\mathbf{Q}L.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L^{wfin}$, $\mathbf{Q}^{=}L.\mathbf{bf}^{wfin}$ *are all* Π^0_1*-complete.*

Corollary 4. *Let* L *be one of the logics* **K**, **T**, **D**, **K4**, **K4.3**, **S4**, **S4.3**, **GL**, **GL.3**, **Grz**, **Grz.3**, **KB**, **KTB**, **K5**, **K45**, **S5**. *Then, the monadic fragments of the logics* $\mathbf{Q}L_n^{wfin}$, $\mathbf{Q}L_n.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L_n^{wfin}$, $\mathbf{Q}^{=}L_n.\mathbf{bf}^{wfin}$ *are* Π^0_1*-complete.*

From Proposition 1 we also obtain the following:

Theorem 5. *Let* $\mathscr{C}$ *be a decidable class of Kripke* 1*-frames closed under the operation of taking subframes and satisfying the condition that there exists* $m \in \mathbb{N}$ *such that* $|R(w)| \leqslant m$ *whenever* $\langle W, R\rangle \in \mathscr{C}$ *and* $w \in W$. *Then the monadic fragments of the logics* $L(\mathscr{C})$, $L_c(\mathscr{C})$ *and the monadic fragments with equality of the logics* $L^{=}(\mathscr{C})$, $L_c^{=}(\mathscr{C})$ *are decidable.*

Recall that $\mathbf{Alt}_n$ is a monomodal logic complete with respect to the class of Kripke frames where every world sees at most n worlds. Using completeness of the predicate counterpart of $\mathbf{Alt}_n$ [16], which, using [3, Theorem 3.8.7], implies the completeness of $\mathbf{Q}^{=}\mathbf{Alt}_n$, we obtain the following:

Theorem 6. *The monadic fragments of logics* $\mathbf{QAlt}_n$, $\mathbf{QAlt}_n.\mathbf{bf}$, $\mathbf{Q}^{=}\mathbf{Alt}_n$, *and* $\mathbf{Q}^{=}\mathbf{Alt}_n.\mathbf{bf}$ *are all decidable.*

4. Discussion

Proposition 1 and Theorem 2 remain true in the Kripke semantics with equality as identity. Proposition 1 and Theorem 2 can be extended to logics of frames with distinguished worlds. All the results remain true if logics $\mathbf{Q}L.\mathbf{bf}$ and $\mathbf{Q}^{=}L.\mathbf{bf}$ are replaced with logics in which, for some k, the Barcan formula $\boldsymbol{bf}_k$ is replaced with $\Box\boldsymbol{bf}_k$, where $\Box$ is a finite sequence of $\Box_1, \ldots, \Box_n$. Lastly, we note that similar results can be obtained for superintuitionistic monadic logics [15].

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