# Computational complexity of theories of residuated structures

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# Basic setting from the logical point of view

- We are thinking of propositional logics specified using Gentzen-style deductive systems whose primary entities are sequents of the form
$$\Gamma \vdash \Delta,$$
where $\Gamma$ and $\Delta$ are structure composed of formulas using a binary non-associative and not necessarily commutative operator, usually denoted by comma (we also need parentheses for the grouping of formulas).
- We naturally want
$$\Gamma \vdash \Gamma \quad \text{and} \quad \Gamma \vdash \Delta \ \&\ \Delta \vdash \Theta \Rightarrow \Gamma \vdash \Theta,$$
i.e., we want $\vdash$ to be reflexive and transitive (we are not necessarily committed to other properties of $\vdash$ such as monotonicity and compactness).

# Basic setting from the logical point of view

- We have binary connectives that internalize in the language structural properties of our sequents:
  - connective $\circ$ ('fusion') represents the comma: if $\gamma_1$ and $\gamma_2$ correspond, respectively, to $\Gamma_1$ and $\Gamma_2$, then $[\gamma_1 \circ \gamma_2] \vdash \Delta$ corresponds to $[\Gamma_1, \Gamma_2] \vdash \Delta$;
  - two connectives $\backslash$ and $/$ internalizing statements about deduction (they differ in whether a designated premise comes from the left or from the right):

$$\gamma_1 \circ \gamma_2 \vdash \delta \iff \gamma_2 \vdash \gamma_1 \backslash \delta;$$
$$\gamma_1 \circ \gamma_2 \vdash \delta \iff \gamma_1 \vdash \delta / \gamma_2.$$

- We might want to have other connectives, say $\wedge$ and $\vee$.
- The basic logic we get is Non-associative Lambek Calculus.
- If we add $\wedge$ and $\vee$ with their usual Gentzen-style rules, we get Full Non-associative Lambek Calculus.
- If, additionally, $\wedge$ and $\vee$ distribute over each other, we get Full Distributive Non-associative Lambek Calculus.

# Residuated ordered groupoids (rogs)

Fix a signature $\sigma$ containing a binary relation symbol $\leqslant$ and binary operational symbols $\circ$, $\backslash$, and $/$.

**Definition**

A ***residuated ordered groupoid*** (for short, ***rog***) is a $\sigma$-structure $\mathbf{A} = \langle A, \circ, \backslash, /, \leqslant \rangle$, where $\langle A, \leqslant \rangle$ is a poset and $\circ$, $\backslash$ and $/$ are binary operations on $A$ such that, for all $a, b, c \in A$,

$$a \circ b \leqslant c \iff b \leqslant a \backslash c \iff a \leqslant c / b. \tag{1}$$

The class of all rogs is denoted by $\mathcal{ROG}$.

# Theories of rogs

The *atomic theory* of $\mathcal{ROG}$ is the set of the atomic formulas (i.e., expressions of the form $s \leqslant t$) valid in $\mathcal{ROG}$. This theory is in $\mathsf{P}$ (E. Aarts and K. Trautwein [1]).

The *Horn theory* of $\mathcal{ROG}$ is the set of formulas of the form $\alpha_1 \dot{\wedge} \ldots \dot{\wedge} \alpha_n \Rightarrow \alpha$, where $\alpha_1, \ldots, \alpha_n$ and $\alpha$ are all atomic, valid in $\mathcal{ROG}$. This theory is in $\mathsf{P}$ (W. Buszkowski [2]).

The *universal theory* of $\mathcal{ROG}$ is the set of formulas $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where $\varphi$ is a Boolean combination of atomic formulas, valid in $\mathcal{ROG}$. This theory is $\mathsf{coNP}$-complete (this talk & *JoLLI* paper).

# Partial structures

**Definition**

A ***partial $\sigma$-structure*** is a tuple $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$, where $B \neq \varnothing$, $\leqslant^{\mathbf{B}} \subseteq B \times B$, and $\circ^{\mathbf{B}}$, $\backslash^{\mathbf{B}}$, and $/^{\mathbf{B}}$ are partial binary operations on $B$ (i.e., partial functions $B \times B \mapsto B$).

The domains of $\circ^{\mathbf{B}}$, $\backslash^{\mathbf{B}}$ and $/^{\mathbf{B}}$ are denoted by, respectively, $\operatorname{dom} \circ^{\mathbf{B}}$, $\operatorname{dom} \backslash^{\mathbf{B}}$, and $\operatorname{dom} /^{\mathbf{B}}$.

# Partial rogs

## Definition

A ***partial rog*** is a partial $\sigma$-structure $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ that is a partial substructure of a rog, i.e., such that there exists a rog $\mathbf{A} = \langle A, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, \leqslant^{\mathbf{A}} \rangle$ with $B \subseteq A$, $\leqslant^{\mathbf{B}} = \leqslant^{\mathbf{A}} \restriction_B$ and $a \star^{\mathbf{B}} b = a \star^{\mathbf{A}} b$ for every $\star \in \{\circ, \backslash, /\}$ and every $\langle a, b \rangle \in \operatorname{dom} \star^{\mathbf{B}}$.

**Caution**: if $\mathbf{B}$ is a partial rog that is a partial substructure of a rog $\mathbf{A}$, then $\star^{\mathbf{B}}$ ($\star \in \{\circ, \backslash, /\}$) is not necessarily a restriction of $\star^{\mathbf{A}}$ to $B$. It is possible that $a, b \in B$ and $a \star^{\mathbf{A}} b \in B$, but $\langle a, b \rangle \notin \operatorname{dom} \star^{\mathbf{B}}$; i.e., we do not require that $\operatorname{dom} \star^{\mathbf{B}} = \operatorname{dom} \star^{\mathbf{A}} \restriction B$.

**E.g.**, we might have $\langle a_1, a_2 \rangle \in \operatorname{dom} \circ^{\mathbf{B}}$, $\langle b_1, b_2 \rangle \in \operatorname{dom} \backslash^{\mathbf{B}}$, and $a_2 \circ^{\mathbf{A}} b_1 = a_1 \circ^{\mathbf{A}} a_2 (= a_1 \circ^{\mathbf{B}} a_2)$, but $\langle a_2, b_1 \rangle \notin \operatorname{dom} \circ^{\mathbf{B}}$.

# Embedding a partial structure into a structure

## Definition

Let $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ be a partial $\sigma$-structure and $\mathbf{A} = \langle A, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, \leqslant^{\mathbf{A}} \rangle$ a $\sigma$-structure. An ***embedding*** of $\mathbf{B}$ into $\mathbf{A}$ is a map $\alpha : B \to A$ such that

- $a \leqslant^{\mathbf{B}} b \iff \alpha(a) \leqslant^{\mathbf{A}} \alpha(b)$, for every $a, b \in B$;
- $\alpha(a \star^{\mathbf{B}} b) = \alpha(a) \star^{\mathbf{A}} \alpha(b)$, for every $\star \in \{\circ, \backslash, /\}$ and every $\langle a, b \rangle \in \mathrm{dom} \star^{\mathbf{B}}$.

## Fact

*If a partial $\sigma$-structure $\mathbf{B}$ is embeddable into a rog $\mathbf{A}$, then $\mathbf{B}$ is isomorphic to a partial substructure of $\mathbf{A}$; hence, $\mathbf{B}$ is a partial rog.*

# Characterization of partial rogs

## Theorem

*A partial σ-structure* $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ *is a partial rog iff the following conditions are satisfied:*

(i) $\langle B, \leqslant^{\mathbf{B}} \rangle$ *is a poset;*

(ii) $\forall \langle a, b \rangle, \langle c, d \rangle \in \mathrm{dom} \circ^{\mathbf{B}} [a \leqslant^{\mathbf{B}} c \;\&\; b \leqslant^{\mathbf{B}} d \Longrightarrow a \circ^{\mathbf{B}} b \leqslant^{\mathbf{B}} c \circ^{\mathbf{B}} d]$;

(iii) $\forall \langle a, b \rangle \in \mathrm{dom} \circ^{\mathbf{B}} \forall \langle c, d \rangle \in \mathrm{dom} \backslash^{\mathbf{B}}$
$[a \leqslant^{\mathbf{B}} c \;\&\; b \leqslant^{\mathbf{B}} c \backslash^{\mathbf{B}} d \Rightarrow a \circ^{\mathbf{B}} b \leqslant^{\mathbf{B}} d]$;

(iv) $\forall \langle a, b \rangle \in \mathrm{dom} \circ^{\mathbf{B}} \forall \langle c, d \rangle \in \mathrm{dom} /^{\mathbf{B}}$
$[a \leqslant^{\mathbf{B}} c /^{\mathbf{B}} d \;\&\; b \leqslant^{\mathbf{B}} d \Rightarrow a \circ^{\mathbf{B}} b \leqslant^{\mathbf{B}} c]$;

(v) $\forall \langle a, b \rangle \in \mathrm{dom} \backslash^{\mathbf{B}} \forall \langle c, d \rangle \in \mathrm{dom} \circ^{\mathbf{B}}$
$[a \leqslant^{\mathbf{B}} c \;\&\; c \circ^{\mathbf{B}} d \leqslant^{\mathbf{B}} b \Rightarrow d \leqslant^{\mathbf{B}} a \backslash^{\mathbf{B}} b]$;

(vi) $\forall \langle a, b \rangle \in \mathrm{dom} /^{\mathbf{B}} \forall \langle c, d \rangle \in \mathrm{dom} \circ^{\mathbf{B}}$
$[b \leqslant^{\mathbf{B}} d \;\&\; c \circ^{\mathbf{B}} d \leqslant^{\mathbf{B}} a \Rightarrow c \leqslant^{\mathbf{B}} a /^{\mathbf{B}} b]$;

# Characterization of partial rogs (contnd)

**Theorem**

*A partial σ-structure* $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ *is a partial rog iff* $\langle B, \leqslant^{\mathbf{B}} \rangle$ *is a poset and the following conditions are satisfied:*

$$\ldots$$

(vii) $\forall \langle a, b\rangle, \langle c, d\rangle \in \mathrm{dom} \backslash^{\mathbf{B}} [a \leqslant^{\mathbf{B}} c \mathbin{\&} d \leqslant^{\mathbf{B}} b \Rightarrow c \backslash^{\mathbf{B}} d \leqslant^{\mathbf{B}} a \backslash^{\mathbf{B}} b]$;

(viii) $\forall \langle a, b\rangle \in \mathrm{dom} \backslash^{\mathbf{B}} \forall \langle c, d\rangle \in \mathrm{dom} /^{\mathbf{B}}$
$[a \leqslant^{\mathbf{B}} c /^{\mathbf{B}} d \mathbin{\&} c \leqslant^{\mathbf{B}} b \Rightarrow d \leqslant^{\mathbf{B}} a \backslash^{\mathbf{B}} b]$;

(ix) $\forall \langle a, b\rangle \in \mathrm{dom} /^{\mathbf{B}} \forall \langle c, d\rangle \in \mathrm{dom} \backslash^{\mathbf{B}}$
$[d \leqslant^{\mathbf{B}} a \mathbin{\&} b \leqslant^{\mathbf{B}} c \backslash^{\mathbf{B}} d \Rightarrow c \leqslant^{\mathbf{B}} a /^{\mathbf{B}} b]$;

(x) $\forall \langle a, b\rangle, \langle c, d\rangle \in \mathrm{dom} /^{\mathbf{B}} [c \leqslant^{\mathbf{B}} a \mathbin{\&} b \leqslant^{\mathbf{B}} d \Rightarrow c /^{\mathbf{B}} d \leqslant^{\mathbf{B}} a /^{\mathbf{B}} b]$.

($\Rightarrow$) The analogues of properties (i) through (x) hold in every rog.

($\Leftarrow$) We construct a relational frame $\mathfrak{F}$ from $\mathbf{B}$ and then a rog $\mathbf{A}_{\mathfrak{F}}$ out of $\mathfrak{F}$, and embed $\mathbf{B}$ into $\mathbf{A}_{\mathfrak{F}}$.

# Relational frames

Relational frames are widely used in the study of non-classical logics, due to the success of the Kripke frame semantics for modal and superintuitionistic logics. The relational frame theory for rogs and related structures is due to Dunn [3].

**Definition**

A ***frame*** is a relational structure $\mathfrak{F} = \langle P, \leqslant, R \rangle$, where $\langle P, \leqslant \rangle$ is a poset and $R$ is a ternary relation on $P$ that is monotone in the last coordinate and antitone in the first two coordinates: for every $f, f', g, g', h, h' \in P$,

$$R(f, g, h) \mathbin{\&} f' \leqslant f \mathbin{\&} g' \leqslant g \mathbin{\&} h \leqslant h' \Longrightarrow R(f', g', h'). \tag{2}$$

# From frames to algebras

Let $\mathfrak{F} = \langle P, \leqslant, R \rangle$ be a frame and $U(P)$ be the set of upsets of $\mathfrak{F}$ (i.e. if $X \in U(P)$, $f \in X$ and $f \leqslant g$, then $g \in X$).
Define, for all $X, Y \in U(P)$,

$$X \circ Y := \{h \in P \mid \exists f, g \in P\, [f \in X \;\&\; g \in Y \;\&\; R(f, g, h)]\}; \qquad (3)$$

$$X \backslash Y := \{g \in P \mid \forall f, h \in P\, [f \in X \;\&\; R(f, g, h) \Rightarrow h \in Y]\}; \qquad (4)$$

$$Y / X := \{f \in P \mid \forall g, h \in P\, [g \in X \;\&\; R(f, g, h) \Rightarrow h \in Y]\}. \qquad (5)$$

Since $\mathfrak{F}$ satisfies (2), so defined $\circ$, $\backslash$ and $/$ are operations on $U(P)$.
The definitions (3)–(5) ensure that (1) is satisfied with respect to $\subseteq$ on $U(P)$. Hence, $\mathbf{A}_{\mathfrak{F}} = \langle U(P), \circ, \backslash, /, \subseteq \rangle$ is a rog.

# From algebras to frames

Let $\mathbf{A} = \langle A, \circ, \backslash, /, \leqslant \rangle$ be a rog. Define a ternary relation $R$ on $U(A)$ by

$$R(f, g, h) \iff \forall a, b \in A\, [a \in f \,\&\, b \in g \Longrightarrow a \circ b \in h]. \tag{6}$$

Then $R$ and $\subseteq$ satisfy condition (2), hence $\mathfrak{F}_{\mathbf{A}} = \langle U(A), \subseteq, R \rangle$ is a frame.

**Fact**

*Let $\mathbf{A} = \langle A, \circ, \backslash, /, \leqslant \rangle$ be a rog. The map $\mu \colon A \to U(U(A))$ defined by $\mu(a) = \{f \in U(A) \mid a \in f\}$ is an embedding of $\mathbf{A}$ into $\mathbf{A}_{\mathfrak{F}_{\mathbf{A}}}$.*

# Proof idea for part ($\Leftarrow$) of the Theorem

Suppose $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ is a partial $\sigma$-structure satisfying (i) through (x). We obtain a rog into which $\mathbf{B}$ is embeddable. Define a ternary relation $R^{\mathbf{B}}$ on $U(B)$ by:

$$\begin{aligned} R^{\mathbf{B}}(f, g, h) \iff & \forall \langle a, b \rangle \in \operatorname{dom} \circ^{\mathbf{B}} [a \in f \ \& \ b \in g \Longrightarrow a \circ^{\mathbf{B}} b \in h] \\ & \& \ (\forall \langle a, b \rangle \in \operatorname{dom} \backslash^{\mathbf{B}} [a \in f \ \& \ a \backslash^{\mathbf{B}} b \in g \Longrightarrow b \in h] \\ & \& \ \forall \langle a, b \rangle \in \operatorname{dom} /^{\mathbf{B}} [a /^{\mathbf{B}} b \in f \ \& \ b \in g \Longrightarrow a \in h]. \end{aligned}$$

Then $\mathfrak{F} = \langle U(B), \subseteq, R^{\mathbf{B}} \rangle$ is a frame.
Let $\mathbf{A}_{\mathfrak{F}} = \langle U(U(B)), \circ, \backslash, /, \subseteq \rangle$ be the rog associated with $\mathfrak{F}$ and let $\mu \colon B \to U(U(B))$ be the map defined by $\mu(a) = \{f \in U(B) \mid a \in f\}$.
Then $\mu$ is an embedding of $\mathbf{B}$ into $\mathbf{A}_{\mathfrak{F}}$.

# Evaluation of formulas in rogs

Universal $\sigma$-sentences are formulas of the form $\forall x_1 \ldots \forall x_n \varphi$, where $\varphi$ is a quantifier-free (first-order) $\sigma$-formula, i.e., a formula defined by the BNF expression

$$\varphi := t \leqslant t \mid \dot{\neg}\varphi \mid (\varphi \mathbin{\dot{\wedge}} \varphi) \mid (\varphi \mathbin{\dot{\vee}} \varphi),$$

with $t$ ranging over $\sigma$-terms, and containing no variables other than $x_1, \ldots, x_n$.

Formulas are evaluated as in standard model theory. The *universal theory of* $\mathcal{ROG}$ is the set of all universal $\sigma$-sentences valid on $\mathcal{ROG}$.

By the semantics of quantifiers, a universal sentence $\forall x_1 \ldots \forall x_n \, \varphi$ is valid on $\mathcal{ROG}$ iff $\dot{\neg}\varphi$ is not satisfiable in $\mathcal{ROG}$. Thus, satisfiability of quantifier-free $\sigma$-formulas in $\mathcal{ROG}$ and membership in the universal theory of $\mathcal{ROG}$ are complementary computational problems.

# Evaluation of quantifier-free formulas in partial rogs

We shall also need the notion of satisfaction of a quantifier-free $\sigma$-formula in a partial rog under a partial assignment (partial function from variables into the universe of a partial rog). Let $\mathbf{B}$ be a partial rog and $v$ a partial assignment in $\mathbf{B}$.

Define the relation $\mathbf{B}\downarrow v(t)$ ("the value of $t$ in $\mathbf{B}$ is defined under $v$"):

$$\begin{array}{lcl} \mathbf{B}\downarrow v(x_i) & \iff & x_i \in \operatorname{dom} v; \\ \mathbf{B}\downarrow v(t_1 \star t_2) & \iff & \mathbf{B}\downarrow v(t_1),\ \mathbf{B}\downarrow v(t_2) \text{ and } \langle v(t_1), v(t_2)\rangle \in \operatorname{dom} \star^{\mathbf{B}}, \\ & & \text{where } \star \in \{\circ, \backslash, /\}. \end{array}$$

Intuitively, $\mathbf{B} \models^v \varphi$ and $\mathbf{B} \not\models^v \varphi$ mean that the relation $\mathbf{B}\downarrow v(t)$ holds for enough terms of $\varphi$ for the value of $\varphi$ in $\mathbf{B}$ under $v$ to come out as, respectively, true and false.

# Evaluation of quantifier-free formulas in partial rogs

Formally, we define the relations $\mathbf{B} \models^v \varphi$ ("$\varphi$ is satisfied in $\mathbf{B}$ under $v$"), $\mathbf{B} \not\models^v \varphi$ ("$\varphi$ is not satisfied in $\mathbf{B}$ under $v$") and $\mathbf{B} \approx^v \varphi$ ("the value of $\varphi$ in $\mathbf{B}$ under $v$ is undefined"):

$$
\begin{array}{lcl}
\mathbf{B} \models^v t_1 \leqslant t_2 & \iff & \mathbf{B}\!\downarrow\! v(t_1),\ \mathbf{B}\!\downarrow\! v(t_2) \text{ and } v(t_1) \leqslant^{\mathbf{B}} v(t_2); \\
\mathbf{B} \not\models^v t_1 \leqslant t_2 & \iff & \mathbf{B}\!\downarrow\! v(t_1),\ \mathbf{B}\!\downarrow\! v(t_2) \text{ and } v(t_1) \not\leqslant^{\mathbf{B}} v(t_2); \\
\mathbf{B} \approx^v t_1 \leqslant t_2 & \text{otherwise;} & \\
\mathbf{B} \models^v \dot{\neg}\varphi & \iff & \mathbf{B} \not\models^v \varphi; \\
\mathbf{B} \not\models^v \dot{\neg}\varphi & \iff & \mathbf{B} \models^v \varphi; \\
\mathbf{B} \approx^v \dot{\neg}\varphi & \text{otherwise;} & \\
\mathbf{B} \models^v \varphi_1 \dot{\wedge} \varphi_2 & \iff & \mathbf{B} \models^v \varphi_1 \text{ and } \mathbf{B} \models^v \varphi_2; \\
\mathbf{B} \not\models^v \varphi_1 \dot{\wedge} \varphi_2 & \iff & \mathbf{B} \not\models^v \varphi_1 \text{ or } \mathbf{B} \not\models^v \varphi_2; \\
\mathbf{B} \approx^v \varphi_1 \dot{\wedge} \varphi_2 & \text{otherwise;} & \\
\mathbf{B} \models^v \varphi_1 \dot{\vee} \varphi_2 & \iff & \mathbf{B} \models^v \varphi_1 \text{ or } \mathbf{B} \models^v \varphi_2; \\
\mathbf{B} \not\models^v \varphi_1 \dot{\vee} \varphi_2 & \iff & \mathbf{B} \not\models^v \varphi_1 \text{ and } \mathbf{B} \not\models^v \varphi_2; \\
\mathbf{B} \approx^v \varphi_1 \dot{\vee} \varphi_2 & \text{otherwise.} &
\end{array}
$$

# Evaluation of quantifier-free formulas in partial rogs

A quantifier-free $\sigma$-formula $\varphi$ is *satisfiable* in a partial rog $\mathbf{B}$ if there exists a partial assignment $v$ on $\mathbf{B}$ such that $\mathbf{B} \models^v \varphi$.

# Measures of complexity of formulas

The standard measure of complexity of a formula $\varphi$ is its length $\mathit{len}\,\varphi$ (the number of occurrences of symbols in $\varphi$).

For us, it's more convenient to work with the following measure:

$$\mathit{size}\,\varphi \quad = \quad \#\text{ of variables} + \#\text{ of occurrences of operation symbols in }\varphi.$$

Surely, $\mathit{size}\,\varphi \leqslant \mathit{len}\,\varphi$, so we are fine.

# Main theorem for rogs

**Lemma**

*A quantifier-free $\sigma$-formula $\varphi$ is satisfiable in $\mathcal{ROG}$ iff it is satisfiable in a partial rog whose cardinality does not exceed size $\varphi$.*

**Proof.**

('only if') Let $\mathbf{A} \models^{v} \varphi$, for a rog $\mathbf{A}$. Put $B = \{v(t) \mid t \in \mathit{terms}\, \varphi\}$. Then $|B| \leqslant \mathit{size}\, \varphi$. For all $a_1, a_2 \in B$ and $\star \in \{\circ, \backslash, /\}$, let $\langle a_1, a_2 \rangle \in \mathrm{dom}(\star^{\mathbf{B}})$ if there exists $t_1 \star t_2 \in \mathit{terms}\, \varphi$ with $a_1 = v(t_1)$ and $a_2 = v(t_2)$. Then, for every $\star \in \{\circ, \backslash, /\}$ and $\langle a_1, a_2 \rangle \in \mathrm{dom}(\star^{\mathbf{B}})$, set $a_1 \star^{\mathbf{B}} a_2 := a_1 \star^{\mathbf{A}} a_2$. Set $\leqslant^{\mathbf{B}} = \leqslant^{\mathbf{A}} \restriction_B$. Then $\mathbf{B} := \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ is a partial rog. Let $\bar{v} := v \restriction_{\mathit{var}\, \varphi}$. Then $\mathbf{B} \models^{\bar{v}} \varphi$. Thus, $\varphi$ is satisfiable in a partial rog of the required cardinality.

('if') Let $\mathbf{B} \models^{\bar{v}} \varphi$, for a partial rog $\mathbf{B}$ and a partial assignment $\bar{v}$. Let $\mathbf{B}$ be a partial substructure of a rog $\mathbf{A}$. Let $v$ be a assignment on $\mathbf{B}$ extending $\bar{v}$. Then, $\mathbf{B} \models^{v} \varphi$. Since $\mathbf{B}$ is a partial substructure of $\mathbf{A}$, it follows that $\mathbf{A} \models^{v} \varphi$. □

# Main theorem for rogs

**Theorem**

*Satisfiability of quantifier-free $\sigma$-formulas in $\mathcal{ROG}$ is in* NP. *Hence, the universal theory of $\mathcal{ROG}$ is in* coNP.

**Proof.**

Let $\varphi$ be a quantifier-free $\sigma$-formula. By Lemma, it is enough to check if it is satisfiable in a partial rog of cardinality $\leqslant$ *size* $\varphi$. We use a nondeterministic algorithm: Guess a partial $\sigma$-structure $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ with $|B| \leqslant$ *size* $\varphi$ and a partial assignment $\bar{v}$ on $\mathbf{B}$. Check whether $\mathbf{B}$ is a partial rog and whether $\mathbf{B} \models^{\bar{v}} \varphi$. If both checks succeed, return "yes"; otherwise, return "no."

In view of Theorem, to check if $\mathbf{B}$ is a partial rog, it is enough to check properties (i) through (x), which can be done in time polynomial in $|B| \leqslant$ *size* $\varphi$. Checking whether $\mathbf{B} \models^{\bar{v}} \varphi$ can also be done in time polynomial in *size* $\varphi$. □

# Main theorem for rogs

We say that a $k$-ary predicate $P$ on a structure with domain $A$ is *non-trivial* if $P \neq \varnothing$ and $P \neq A^k$; we say that a structure is *non-trivial* if it has a non-trivial predicate definable in its signature.

**Proposition**

*Let $\mathcal{K}$ be a class of structures containing a non-trivial structure. Then, satisfiability of quantifier-free first-order formulas in $\mathcal{K}$ is* NP*-hard and, hence, the universal theory of $\mathcal{K}$ is* coNP*-hard.*

**Proof.**

Reduction from SAT. Use non-triviality to simulate Boolean variables. □

**Theorem**

*Satisfiability of quantifier-free $\sigma$-formulas in $\mathcal{ROG}$ is* NP*-complete. Hence, the universal theory of $\mathcal{ROG}$ is* coNP*-complete.*

# Unital and integral rogs

Let $\sigma^1$ be an expansion of signature $\sigma$ with a constant 1.

**Definition**

A ***unital rog*** (for short, ***urog***) is a $\sigma^1$-structure $\mathbf{A} = \langle A, \circ, \backslash, /, \mathbf{1}, \leqslant \rangle$, where $\langle A, \circ, \backslash, /, \leqslant \rangle$ is a rog and $\mathbf{1} \in A$ such that $a \circ \mathbf{1} = a = \mathbf{1} \circ a$, for every $a \in A$.

**Definition**

An ***integral rog*** (for short, ***irog***) is a urog where $a \leqslant \mathbf{1}$, for every $a \in A$.

Using techniques similar to those used for rogs, we obtain the following:

**Theorem**

*Satisfiability of quantifier-free $\sigma^1$-formulas both in urogs and irogs is* NP*-complete. Hence, the universal theories of urogs and irogs are both* coNP*-complete.*

# Residuated algebras

**Definition**

Let $k \geqslant 1$. A ***residuated $k$-algebra*** is a structure $\mathbf{A} = \langle A, \mathbf{t}, \mathbf{r}_1, \ldots, \mathbf{r}_k, \leqslant \rangle$, where $\langle A, \leqslant \rangle$ is a poset and $\mathbf{A}$ satisfies the $k$-ary residuation property: for every $a_1, \ldots, a_k, c \in A$ and every $j \in \{1, \ldots, k\}$,

$$\mathbf{t}(a_1, \ldots, a_k) \leqslant c \iff a_j \leqslant \mathbf{r}_j(a_1, \ldots, a_{j-1}, c, a_{j+1}, \ldots, a_k). \tag{7}$$

**Definition**

A ***residuated algebra*** is a structure $\mathbf{A} = \langle A, \rho, \leqslant \rangle$, where $\langle A, \leqslant \rangle$ is a poset and $\rho$ is a family of $k$-tuples $\langle \mathbf{t}, \mathbf{r}_1, \ldots, \mathbf{r}_k \rangle$, with $k \geqslant 1$, such that each structure $\mathbf{A} = \langle A, \mathbf{t}, \mathbf{r}_1, \ldots, \mathbf{r}_k, \leqslant \rangle$ is a residuated $k$-algebra.

**Theorem**

*Let $\mathcal{C}$ be a class of residuated algebras. Satisfiability of quantifier-free formulas in $\mathcal{C}$ is* NP*-complete. Hence, the universal theory of $\mathcal{C}$ is* coNP*-complete.*

# Residuated distributive lattice-oriented groupoids (brdgs)

A residuated distributive lattice-oriented groupoid is a rog where the partial order is a distributive lattice. We shall assume, for convenience, that the lattice is bounded.

Fix a signature $\sigma^{brdg}$ containing a binary relation symbol $\leqslant$, binary operational symbols $\wedge$, $\vee$, $\circ$, $\backslash$, $/$, and constants 0 and 1.

**Definition**

A ***bounded residuated distributive lattice-oriented groupoid*** (for short, ***brdg***) is a $\sigma^{brdg}$-structure $\mathbf{A} = \langle A, \wedge, \vee, \circ, \backslash, /, \leqslant, 0, 1\rangle$, where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, $\leqslant$ is the partial order associated with the lattice, and $\circ$, $\backslash$ and $/$ are binary operations on $A$ such that, for all $a, b, c \in A$, the residuation condition (1) is satisfied.

The class of all brdgs is denoted by $\mathcal{BRDG}$.

Inequality is defined in the usual way: $a \leqslant b \ := \ a \wedge b = a$.

# Theories of brdgs

The *equational theory* of $\mathcal{BRDG}$ is the set of equations valid in $\mathcal{BRDG}$. (Conjecture: this theory is coNP-complete).

The *quasi-equational theory* of $\mathcal{BROG}$ is the set of quasi-equations valid in $\mathcal{BRDG}$. This theory is EXPTIME-complete (this talk & *Algebra Universalis* paper).

The *universal theory* of $\mathcal{BRDG}$ is the set of formulas $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where $\varphi$ is a Boolean combination of atomic formulas, valid in $\mathcal{BRDG}$. This theory is EXPTIME-complete (this talk & *Algebra Universalis* paper).

# Relational frames

## Definition (Recall)

A ***frame*** is a relational structure $\mathfrak{F} = \langle P, \leqslant, R \rangle$, where $\langle P, \leqslant \rangle$ is a poset and $R$ is a ternary relation on $P$ that is monotone in the last coordinate and antitone in the first two coordinates: for every $f, f', g, g', h, h' \in P$,

$$R(f, g, h) \ \& \ f' \leqslant f \ \& \ g' \leqslant g \ \& \ h \leqslant h' \Longrightarrow R(f', g', h').$$

# From frames to algebras and back

Let $\mathfrak{F} = \langle P, \leqslant, R \rangle$ be a frame and $U(P)$ be the set of upsets of $\mathfrak{F}$. Define operations on $U(P)$ as before, i.e., by (3)–(5). Then, $\mathbf{A}_{\mathfrak{F}} = \langle U(P), \cap, \cup, \circ, \backslash, /, \subseteq, \varnothing, P \rangle$ is a brdg.

Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \backslash, /, \leqslant, 0, 1 \rangle$ be a brdg and let $P(A)$ be the set of prime filters of $\mathbf{A}$. Define a ternary relation $R$ on by (2):

$$R(f, g, h) \iff \forall a, b \in A\, [a \in f \ \& \ b \in g \Longrightarrow a \circ b \in h].$$

Then $R$ and $\subseteq$ satisfy condition (2), hence $\mathfrak{F}_{\mathbf{A}} = \langle P(A), \subseteq, R \rangle$ is a frame.

**Fact**

*Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \backslash, /, \leqslant, 0, 1 \rangle$ be a brdg. The map $\mu \colon A \to U(P)$ defined by $\mu(a) = \{f \in P \mid a \in f\}$ is an embedding of $\mathbf{A}$ into $\mathbf{A}_{\mathfrak{F}_{\mathbf{A}}}$.*

# Partial $\sigma^{brdg}$-structures and partial rdgs

## Definition

A ***partial*** $\boldsymbol{\sigma^{brdg}}$***-structure*** is a tuple $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$, where $B \neq \varnothing$, $\leqslant^{\mathbf{B}} \subseteq B \times B$, $0^{\mathbf{B}}, 1^{\mathbf{B}} \in B$, and $\wedge^{\mathbf{B}}$, $\vee^{\mathbf{B}}$, $\circ^{\mathbf{B}}$, $\backslash^{\mathbf{B}}$, and $/^{\mathbf{B}}$ are partial binary operations on $B$ (i.e., partial functions $B \times B \mapsto B$).

## Definition

A ***partial brdg*** is a partial $\sigma^{brdg}$-structure $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ that is a partial substructure of a brdg, i.e., such that there exists a brdg $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, \leqslant^{\mathbf{A}} \rangle$ with $B \subseteq A$, $\leqslant^{\mathbf{B}} = \leqslant^{\mathbf{A}}\restriction_B$, $0^{\mathbf{B}} = 0^{\mathbf{A}}$, $1^{\mathbf{B}} = 1^{\mathbf{A}}$, and $a \star^{\mathbf{B}} b = a \star^{\mathbf{A}} b$, for every $\star \in \{\wedge, \vee, \circ, \backslash, /\}$ and every $\langle a, b \rangle \in \mathrm{dom} \star^{\mathbf{B}}$.

# Embedding a partial structure into a structure

**Definition**

Let $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}} \rangle$ be a partial $\sigma^{brdg}$-structure and $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, \leqslant^{\mathbf{A}} \rangle$ a $\sigma^{brdg}$-structure. An ***embedding*** of $\mathbf{B}$ into $\mathbf{A}$ is a map $\alpha : B \to A$ such that

- $a \leqslant^{\mathbf{B}} b \iff \alpha(a) \leqslant^{\mathbf{A}} \alpha(b)$, for every $a, b \in B$;
- $\alpha(0^{\mathbf{B}}) = 0^{\mathbf{A}}$;
- $\alpha(1^{\mathbf{B}}) = 1^{\mathbf{A}}$;
- $\alpha(a \star^{\mathbf{B}} b) = \alpha(a) \star^{\mathbf{A}} \alpha(b)$, for every $\star \in \{\wedge, \vee, \circ, \backslash, /\}$ and every $\langle a, b \rangle \in \mathrm{dom} \star^{\mathbf{B}}$.

**Fact**

*If a partial $\sigma^{brdg}$-structure $\mathbf{B}$ is embeddable into a brdg $\mathbf{A}$, then $\mathbf{B}$ is isomorphic to a partial substructure of $\mathbf{A}$; hence, $\mathbf{B}$ is a partial brdg.*

# Characterization of partial bounded lattices

Fix the signature $\sigma^{b\ell}$ containing $\wedge$, $\vee$, 0, and 1.

**Theorem (Van Alten 2013)**

*A partial $\sigma^{b\ell}$-structure $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ is a partial bounded lattice if $\leqslant^{\mathbf{B}}$ is a partial order on $B$, with bounds $0^{\mathbf{B}}$ and $1^{\mathbf{B}}$, and $\wedge^{\mathbf{B}}$ and $\vee^{\mathbf{B}}$ are compatible with $\leqslant^{\mathbf{B}}$, i.e.,*

- *if $\langle a, b\rangle \in \mathrm{dom} \wedge^{\mathbf{B}}$, then $a \wedge^{\mathbf{B}} b$ is the glb w.r.t. $\leqslant^{\mathbf{B}}$;*
- *if $\langle a, b\rangle \in \mathrm{dom} \vee^{\mathbf{B}}$, then $a \vee^{\mathbf{B}} b$ is the lub w.r.t. $\leqslant^{\mathbf{B}}$.*

# Characterization of partial bounded distributive lattices

**Definition**

Let $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ be a partial lattice. A set $f \subseteq B$ is a ***prime filter*** in $\mathbf{B}$ if the following hold:

- $0^{\mathbf{B}} \notin f$ and $1^{\mathbf{B}} \in f$;
- if $a \in f$ and $a \leqslant^{\mathbf{B}} b$, then $b \in f$;
- if $a \in f$, $b \in f$, and $\langle a, b \rangle \in \operatorname{dom} \wedge^{\mathbf{B}}$, then $a \wedge^{\mathbf{B}} b \in f$;
- if $a \notin f$, $b \notin f$, and $\langle a, b \rangle \in \operatorname{dom} \wedge^{\mathbf{B}}$, then $a \vee^{\mathbf{B}} b \notin f$.

**Theorem (Van Alten 2013)**

*A partial $\sigma^{b\ell}$-structure $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ is a partial bounded distributive lattice if $\mathbf{B}$ is a partial bounded lattice and, moreover, there exists a set $F$ of prime filters of $\mathbf{B}$ such that*

$$\forall a, b \in B \, [a \not\leqslant^{\mathbf{B}} b \Rightarrow \exists f \in F \, (a \in f \;\&\; b \notin F)]. \tag{8}$$

# Characterization of partial brdgs

**Theorem**

*A partial $\sigma^{brdg}$-structure $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ is a partial brdg iff its $\sigma^{b\ell}$-reduct is a partial bounded lattice and there exists a set $\mathcal{F}$ of prime filters of $\mathbf{B}$ such that (8) holds and, moreover,*

$$\begin{array}{ll} \forall h \in F \, \forall \langle a, b \rangle \in \mathrm{dom} \circ^{\mathbf{B}} & [a \circ^{\mathbf{B}} b \in h \Rightarrow \exists f, g \in F(a \in f \ \& \ b \in g \ \& \ R^{\mathbf{B}}(f, g, h))]; \\ \forall g \in F \, \forall \langle a, b \rangle \in \mathrm{dom} \backslash^{\mathbf{B}} & [a \backslash^{\mathbf{B}} b \notin g \Rightarrow \exists f, h \in F(a \in f \ \& \ b \notin h \ \& \ R^{\mathbf{B}}(f, g, h)]; \\ \forall f \in F \, \forall \langle a, b \rangle \in \mathrm{dom} /^{\mathbf{B}} & [a /^{\mathbf{B}} b \notin f \Rightarrow \exists g, h \in F a \in g \ \& \ b \notin h \ \& \ R^{\mathbf{B}}(f, g, h)], \end{array}$$

*where*

$$\begin{array}{rcl} R^{\mathbf{B}}(f, g, h) & \leftrightharpoons & \forall \langle a, b \rangle \in \mathrm{dom} \circ^{\mathbf{B}} (a \in f \ \& \ b \in g \Rightarrow a \circ^{\mathbf{B}} b \in h) \ \& \\ & & \forall \langle a, b \rangle \in \mathrm{dom} \backslash^{\mathbf{B}} (a \in f \ \& \ a \backslash^{\mathbf{B}} b \in g \Rightarrow b \in h) \ \& \\ & & \forall \langle a, b \rangle \in \mathrm{dom} /^{\mathbf{B}} (b /^{\mathbf{B}} a \in f \ \& \ a \in g \Rightarrow b \in h). \end{array}$$

# Characterization of partial brdgs (contd)

**Proof.**

('only if') Let $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ be a partial substructure of a brdg $\mathbf{A}$. Then, $\langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ is a partial bounded lattice. We need to exhibit a set of filters satisfying (8). Set $F := \{\mathcal{F} \cap B \mid \mathcal{F} \text{ is a prime filter of } \mathbf{A}\}$. It can be shown that $F$ is the required set of prime filters.

('if') Let $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, \circ^{\mathbf{B}}, \backslash^{\mathbf{B}}, /^{\mathbf{B}}, \leqslant^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ be a partial $\sigma^{brdg}$-structure satisfying the requirements of the theorem. The structure $\mathfrak{F} = \langle F, \subseteq, R^{\mathbf{B}} \rangle$ is a frame. Let $\mathbf{A}_{\mathfrak{F}} = \langle U(F), \cap, \cup, \circ, \backslash, /, \subseteq, \varnothing, F \rangle$ be the brdg for $\mathfrak{F}$. Define the map $\mu : B \to U(F)$ by $\mu(a) := \{f \in F \mid a \in f\}$. It can be shown that $\mu$ is an embedding of $\mathbf{B}$ into $\mathbf{A}_{\mathfrak{F}}$. Hence, $\mathbf{B}$ is a partial brdg. □

# Upper bound for brdgs

**Lemma**

*A quantifier-free $\sigma^{brdg}$-formula $\varphi$ is satisfiable in $\mathcal{BRDG}$ iff it is satisfiable in a partial brdg whose cardinality does not exceed size $\varphi + 2$.*

**Theorem**

*Satisfiability of quantifier-free $\sigma^{brdg}$-formulas in $\mathcal{BRDG}$ is in* EXPTIME. *Hence, the universal theory of $\mathcal{BRDG}$ is in* EXPTIME.

# Upper bound for brdgs

**Proof.**

Let $\varphi$ be a quantifier-free $\sigma^{brdg}$-formula. By Lemma, it is enough to check if it is satisfiable in a partial brdg of cardinality $\leqslant \mathit{size}\,\varphi + 2$.
We use the following deterministic algorithm to check if a partial $\sigma^{brdg}$-structure $\mathbf{B}$ is a partial brdg:

(1) Check that $\leqslant^{\mathbf{B}}$ is a partial order on $B$, that $0^{\mathbf{B}}$ and $1^{\mathbf{B}}$ are bounds, and that $\wedge^{\mathbf{B}}$ and $\vee^{\mathbf{B}}$ are compatible with $\leqslant^{\mathbf{B}}$ (polynomial);

(2) Check if there exists a set of prime filters of $\mathbf{B}$ with the required properties. To that end,

- Generate all prime filters of $\mathbf{B}$ (exponential in $|\mathbf{B}|$);
- Repeatedly eliminate filters not meeting the desired properties (exponential in $|\mathbf{B}|$);
- If the resultant set is empty, return 'no'; otherwise, check (8).

Using the outlined algorithm, we check all the structures $\sigma^{brdg}$-structures of size $\leqslant \mathit{size}\,\varphi$ to see if they are partial brdgs and, if so, check if $\varphi$ is satisfied there under some partial assignment. □

# Lower bound for brdgs

By reduction from a set of modal formulas describing an $n \times n$ tiling problem through the universal theory of bounded distributive lattices with a unary operator.

**Theorem**

*Satisfiability of quantifier-free $\sigma^{brdg}$-formulas in $\mathcal{BRDG}$ is* EXPTIME*-complete. Hence, the universal theory of $\mathcal{BRDG}$ is* EXPTIME*-complete.*

Since the negation of a formula obtained through the reduction is a quasi-equation, we also obtain the following:

**Theorem**

*The quasi-equational theory of $\mathcal{BRDG}$ is* EXPTIME*-complete.*

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# Thank you!