Semiproducts, products, and modal predicate logics: some examples*

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Abstract

We study two kinds of combined modal logics, semiproducts and products with **S5**, and their correlation with modal predicate logics. We present examples of propositional modal logics for which semiproducts or products with **S5** are axiomatized in the minimal way (they are called semiproduct- or product-matching with **S5**) and also present counterexamples for these properties. The finite model property for (semi)products, together with (semi)product-matching, allow us to show decidability of corresponding 1-variable modal predicate logics.

Keywords: semiproducts of modal logics, products of modal logics, predicate modal logic.

1 Introduction

We study two kinds of combined modal logics, products and semiproducts (also known as expanding products). Products were introduced in the 1970s to formalise reasoning about multiple independent modalities [1, 2]; a comprehensive investigation of the field was undertaken in [3]; some later developments were presented in [4].

So far the study of semiproducts has been lagging behind the study of products. For example, while there is a general theorem on product-matching [3,

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Theorem 5.9], an analogous theorem for semiproducts [3, Theorem 9.10] is much weaker. In many cases, properties of semiproducts are unknown.

In this paper, we are interested in semiproducts and products with **S5**. Our interest is primarily motivated by their connection with modal predicate logics, noted by Fischer-Servi [5]. This connection is, in fact, a bimodal version of the well-known M. Wajsberg's interpretation of **S5** as a single-variable classical predicate logic (see, e.g., [3, Subsection 1.3]).

To study axiomatization of these semiproducts and products and the relationship of these logics with 1-variable modal predicate logics, we introduce a classification of propositional modal logics and give examples for some categories of the proposed classification.

In particular, we consider logics of finite depth, in the sense of [6], with the axiom of thickness corresponding to the Horn condition

$$\forall x, y, z, t\, (xRy \wedge xRz \wedge yRt \to zRt).$$

We show that semiproducts and products of such logics with **S5** are axiomatized in the minimal way and are decidable. Moreover, they have the product and semiproduct finite model property. This implies decidability and the finite model property for corresponding 1-variable predicate logics.

2 Preliminaries

2.1 Propositional modal logics

We consider N-modal propositional formulas constructed from a countable set $PL = \{p_1, p_2, p_3, \dots\}$ of proposition letters, the constant $\perp$, the connective $\to$, and unary modalities $\Box_1, \dots, \Box_N$. In this paper, $N \in \{1, 2\}$.

We use lowercase letters $p, q, r, \dots$ for proposition letters and uppercase letters $A, B, C, \dots$ for formulas. We use the standard abbreviations $\top$, $\neg A$, $A \wedge B$, $A \vee B$, $A \leftrightarrow B$, $\Diamond_i A$ and the iterated modalities $\Box_i^n$ and $\Diamond_i^n$. The modality of the 1-modal language is usually denoted by $\Box$.

A *k-formula* is a formula containing only proposition letters from the set $\{p_1, p_2, \dots, p_k\}$. A 0-formula (i.e., a formula without propositional letters) is called *closed*.

An *N-modal propositional logic* is a set of N-modal formulas containing the Boolean tautologies and formulas of the form $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ and closed under Substitution, Modus Ponens, and Necessitation. The smallest such logic is called $\mathbf{K}_N$; also, $\mathbf{K} := \mathbf{K}_1$.

If $\mathbf{\Lambda}$ is an N-modal logic and A an N-modal formula, then $\mathbf{\Lambda} \vdash A$ means the same as $A \in \mathbf{\Lambda}$. The smallest logic including a logic $\mathbf{\Lambda}$ and a set Γ of formulas is denoted by $\mathbf{\Lambda} + \Gamma$; we write $\mathbf{\Lambda} + A$ instead of $\mathbf{\Lambda} + \{A\}$.

The *fusion* $\mathbf{\Lambda}_1 * \mathbf{\Lambda}_2$ of 1-modal logics $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ is the $\mathbf{K}_2 + \mathbf{\Lambda}_1 \cup \mathbf{\Lambda}_2^{+1}$, where $\mathbf{\Lambda}_2^{+1}$ is obtained from $\mathbf{\Lambda}_2$ by replacing every occurrence of $\Box_1$ with $\Box_2$.

We use standard definitions from Kripke semantics. An N-*frame* is a tuple $F = (W, R_1, \ldots, R_N)$ where $W \neq \varnothing$ and $R_1, \ldots, R_N \subseteq W^2$; elements of W are called *points*. A *Kripke model over* F is a pair $M = (F, \theta)$ where $\theta \colon PL \to 2^W$. The truth relation between points w of a modal M and formulas is defined by recursion; in particular,

- $M, w \models p_i$ if $w \in \theta(p_i)$;
- $M, w \models \Box_i A_1$ if $M, w' \models A_1$ whenever $w R_i w'$.

A formula A is (globally) *true in a model* M (in symbols, $M \vDash A$) if $M, w \vDash A$, for every $w \in W$. A formula A is *valid on* a frame F (in symbols, $F \vDash A$) if $M \vDash A$, for every model M over F.

If Γ is a set of formulas, $\mathbf{V}(\Gamma)$ denotes the class of frames validating Γ; if A is a formula, we write $\mathbf{V}(A)$ instead of $\mathbf{V}(\{A\})$. If $\mathbf{\Lambda}$ is a logic, then $\mathbf{V}(\mathbf{\Lambda})$ is said to be the class of $\mathbf{\Lambda}$-*frames*.

By soundness theorem, $\mathbf{V}(\Gamma) = \mathbf{V}(\mathbf{K}_N + \Gamma)$. Also, if F is an N-frame and $\mathcal{C}$ a class of N-frames, then $\mathbf{L}(F) := \{A \mid F \vDash A\}$ and $\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(F) \mid \boldsymbol{F} \in \mathcal{C}\}$ are N-modal logics. We say that the logic $\mathbf{L}(\mathcal{C})$ is *determined by* $\mathcal{C}$.

A logic is *Kripke complete* if it is determined by some class of frames. A logic has the *finite model property* (fmp), if it is determined by a class of finite frames.

Lemma 2.1. *Let $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ be 1-modal logics and let $F = (W, R_1, R_2)$ be a Kripke frame. Then,*

$$F \vDash \mathbf{\Lambda}_1 * \mathbf{\Lambda}_2 \quad \Longleftrightarrow \quad (W, R_1) \vDash \mathbf{\Lambda}_1 \ \ \& \ \ (W, R_2) \vDash \mathbf{\Lambda}_2.$$

A frame $(W, R_1, \ldots, R_N)$ can also be viewed as a classical first-order model in the signature $\{R_1, \ldots, R_N, =\}$.

Definition 2.2. *A modal logic $\mathbf{\Lambda}$ is* elementary *if the class $\mathbf{V}(\mathbf{\Lambda})$ is definable by a classical first-order sentence. An N-modal formula A and a classical first-order sentence Φ in the signature $\{R_1, \ldots, R_N, =\}$ are* correspondents *if the class $\mathbf{V}(A)$ is definable by Φ.*

Definition 2.3. *A* universal Horn sentence *is a classical first-order sentence in the signature* $\{R_1, \ldots, R_N, =\}$ *of the form*

$$\forall x\, \forall y\, \forall \bar{z}\, (\Phi(x, y, \bar{z}) \to R_i(x, y)),$$

where $\Phi(x, y, \bar{z})$ *is a conjunction of atomic formulas.*

An N*-modal formula* A *is* Horn *if it corresponds to a universal Horn sentence.*

Definition 2.4. *A 1-modal logic* $\mathbf{\Lambda}$ *is* Horn axiomatizable *if* $\mathbf{\Lambda} = \mathbf{K} + \Gamma$*, for some set* Γ *of formulas that are either Horn or closed.*

Definition 2.5. *A* cone *of a frame* $F = (W, R_1, \ldots, R_N)$ *at a point* w*, denoted by* $F{\uparrow}w$*, is the restriction of* F *to the set* $(R_1 \cup \ldots \cup R_N)^*(w)$*, where* S^* *denotes the reflexive transitive closure of a binary relation* S*.*

If $F = F{\uparrow}w$*, then* F *is said to be* rooted at w.

The following is well known:

Lemma 2.6. *Let* F *be a Kripke frame with a set* W *of points. Then,*

$$\mathbf{L}(F) \quad = \quad \bigcap_{w \in W} \mathbf{L}(F{\uparrow}w).$$

Definition 2.7. *A 1-frame* (W, R) *is* n-transitive *if* $R^{n+1} \subseteq \bigcup\limits_{m \leqslant n} R^m$*. An* N*-frame* $(W, R_1, \ldots, R_N)$ *is* n-transitive *if the 1-frame* $(W, R_1 \cup \ldots \cup R_N)$ n*-transitive.*

Let $F = (W, R_1 \cup \ldots \cup R_N)$ be an N-frame and $R = R_1 \cup \ldots \cup R_N$. Note that the points from $R^*(w)$ are path-accessible from w. If F is n-transitive, then all these points are accessible from w in at most n steps, i.e.,

$$R^*(w) \quad = \quad \bigcup_{m \leqslant n} R^m(w).$$

Definition 2.8. *A p-morphism from an* N*-frame* $(W, R_1, \ldots, R_N)$ *onto an* N*-frame* $(W', R'_1, \ldots, R'_N)$ *is a surjective map* $f : W \longrightarrow W'$ *satisfying the following conditions:*

- $xR_iy \Rightarrow f(x)R'_if(y)$ *(lift property),*
- $f(x)R'_iz \Rightarrow \exists y\, (f(y) = z \,\&\, xR_iy)$ *(monotonicity).*

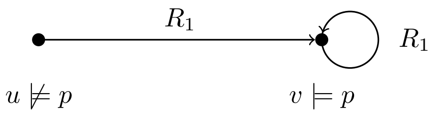

Figure 1. Frame F_0 (with universal R_2) and model M_0

We consider the following 1-modal formulas and logics (here, $n \geqslant 1$):

$$\begin{array}{rclrcl}
\mathit{det} & := & \Diamond p \leftrightarrow \Box p; & \mathit{ref} & := & \Box p \to p; \\
\mathit{sym} & := & \Diamond\Box p \to p; & \mathit{4} & := & \Box p \to \Box\Box p; \\
\mathit{5} & := & \Diamond\Box p \to \Box p; & \mathit{alt}_n & = & \neg \bigwedge\limits_{0\leqslant i\leqslant n} \Diamond(p_i \wedge \bigwedge\limits_{j\neq i} \neg p_j); \\
\mathit{Ath} & := & \Diamond\Diamond p \to \Box\Diamond p. & & &
\end{array}$$

$$\begin{array}{lcllcl}
\mathbf{T} & := & \mathbf{K} + \mathit{ref}; & \mathbf{K4} & := & \mathbf{K} + \mathit{4}; \\
\Box\cdot\mathbf{T} & := & \mathbf{K} + \Box \mathit{ref}; & \mathbf{SL4} & := & \mathbf{K4} + \mathit{det}; \\
\mathbf{K5} & := & \mathbf{K} + \mathit{5}; & \mathbf{K45} & := & \mathbf{K4} + \mathit{5}; \\
\mathbf{S5} & := & \mathbf{K4} + \mathit{ref} + \mathit{sym}; & \mathbf{Alt}_n & := & \mathbf{K} + \mathit{alt}_n; \\
\mathbf{K05} & := & \mathbf{K} + \mathit{Ath}. & & &
\end{array}$$

We briefly mention the Kripke semantics of the lesser known of these logics. The logic $\Box\cdot\mathbf{T}$ is determined by the class of frames satisfying $\forall x \forall y(xRy \to yRy)$. The logic **SL4** is determined by the class of transitive and functional frames, and hence by a single frame where an irreflexive point sees a reflexive point (see Fig. 1). The logic $\mathbf{Alt}_n$ is determined by the frames (W, R) where $|R(w)| \leqslant n$ whenever $w \in W$.

Definition 2.9. *A 1-frame (W, R) is* thick *if $R^{-1} \circ R^2 \subseteq R$ or, equivalently,*

$$\forall x, y, z, u\, (xRy \,\&\, xRz \,\&\, yRu \Rightarrow zRu).$$

Lemma 2.10. *The class $\mathbf{V}(\mathit{Ath})(= \mathbf{V}(\mathbf{K05}))$ is the class of thick 1-frames. The logic* **K05** *is Kripke complete; hence, it is determined by the class of thick 1-frames.*

2.2 Products and semiproducts

Definition 2.11. *The* product *of 1-frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ is the 2-frame $F_1 \times F_2 = (W_1 \times W_2, R_h, R_v)$, where*

$$\begin{array}{lcl}
(x, y)R_h(x', y') & \iff & xR_1x' \quad \& \quad y = y'; \\
(x, y)R_v(x', y') & \iff & x = x' \quad \& \quad yR_2y'.
\end{array}$$

A semiproduct *of F_1 and F_2 is a restriction of $F_1 \times F_2$ to some $W \subseteq W_1 \times W_2$ such that $R_h(W) \subseteq W$ (i.e., W is* horizontally closed*).*

Lemma 2.12. *If F is a semiproduct of F_1 and F_2, and x_i is a point of F_i (here, $i = 1, 2$), then $F{\uparrow}(x_1, x_2)$ is a semiproduct of $F_1{\uparrow}x_1$ and $F_2{\uparrow}x_2$.*

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of 1-frames, then we define

$$\begin{aligned}\mathcal{C}_1 \times \mathcal{C}_2 &:= \{F_1 \times F_2 \mid F_1 \in \mathcal{C}_1 \text{ and } F_2 \in \mathcal{C}_2\},\\ \mathcal{C}_1 \leftthreetimes \mathcal{C}_2 &:= \{F \mid F \text{ is a semiproduct of some frames } F_1 \in \mathcal{C}_1 \text{ and } F_2 \in \mathcal{C}_2\}.\end{aligned}$$

Definition 2.13. *The* product $\mathbf{\Lambda}_1 \times \mathbf{\Lambda}$ *and the* semiproduct $\mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2$ *of 1-modal propositional logics $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are defined as follows:*

$$\begin{aligned}\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2 &:= \mathbf{L}(\mathbf{V}(\mathbf{\Lambda}_1) \times \mathbf{V}(\mathbf{\Lambda}_2));\\ \mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2 &:= \mathbf{L}(\mathbf{V}(\mathbf{\Lambda}_1) \leftthreetimes \mathbf{V}(\mathbf{\Lambda}_2)).\end{aligned}$$

We will make use of the following 2-modal formulas and their frame correspondents:

$$\begin{array}{lll}(chr) & \Diamond_2\Box_1 p \to \Box_1\Diamond_2 p & R_2^{-1} \circ R_1 \subseteq R_1 \circ R_2^{-1};\\ (lcom) & \Box_1\Box_2 p \to \Box_2\Box_1 p & R_2 \circ R_1 \subseteq R_1 \circ R_2;\\ (rcom) & \Box_2\Box_1 p \to \Box_1\Box_2 p & R_1 \circ R_2 \subseteq R_2 \circ R_1.\end{array}$$

Definition 2.14. *We define the* semicommutator $\mathbf{\Lambda}_1 \lrcorner \mathbf{\Lambda}_2$ *and the* commutator $[\mathbf{\Lambda}_1, \mathbf{\Lambda}_2]$ *of 1-modal logics $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ as follows:*

$$\begin{aligned}\mathbf{\Lambda}_1 \lrcorner \mathbf{\Lambda}_2 &:= \mathbf{\Lambda}_1 * \mathbf{\Lambda}_2 + chr + lcom;\\ [\mathbf{\Lambda}_1, \mathbf{\Lambda}_2] &:= \mathbf{\Lambda}_1 \lrcorner \mathbf{\Lambda}_2 + rcom.\end{aligned}$$

Lemma 2.15. *Let $\mathbf{\Lambda}$, $\mathbf{\Lambda}_1$, and $\mathbf{\Lambda}_2$ be 1-modal logics. Then,*

(1) $\mathbf{\Lambda}_1 \lrcorner \mathbf{\Lambda}_2 \subseteq [\mathbf{\Lambda}_1, \mathbf{\Lambda}_2] \subseteq \mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2$.

(2) $\mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2 \subseteq \mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2$.

(3) $\mathbf{\Lambda} \lrcorner \mathbf{S5} \subseteq \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.

(4) $\mathbf{\Lambda} \lrcorner \mathbf{S5} = \mathbf{\Lambda} * \mathbf{S5} + lcom = \mathbf{\Lambda} * \mathbf{S5} + chr$.

Definition 2.16. *A semiproduct logic $\mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2$ has the* semiproduct fmp *if it is determined by a class of finite semiproduct frames. The* product fmp *is defined similarly.*

Remark 2.17. Obviously, the (semi)product fmp implies the fmp. The converse is not always true.

Definition 2.18. *1-modal logics $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are* product-matching *if* $\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2 = [\mathbf{\Lambda}_1, \mathbf{\Lambda}_2]$ *and* semiproduct-matching *if* $\mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2 = \mathbf{\Lambda}_1 \lrcorner \mathbf{\Lambda}_2$.

The following is well known:

Theorem 2.19. *[3, Theorem 5.9]. If a logic* $\mathbf{\Lambda}$ *is Kripke complete and Horn axiomatizable, then* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are product-matching.*

Theorem 2.20. *[3, Theorem 9.10]. If* $\mathbf{\Lambda} \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}\}$*, then* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct-matching.*

Note that, while Theorem 2.19 gives infinitely many examples of product-matching logics, Theorem 2.20 gives only four examples of semiproduct-matching logics.

2.3 Monadic modal predicate logics

We refer to monadic fragments of 1-modal predicate logics as *monadic modal predicate logics*. These are logics in the language containing a countable set $\{x_1, x_2, x_3, \ldots\}$ of individual variables, a countable set $\{P_1^1, P_2^1, P_3^1, \ldots\}$ of monadic predicate letters, a countable set $\{P_1^0, P_2^0, P_3^0, \ldots\}$ of nullary predicate letters (i.e., proposition letters), and logical symbols $\perp$, $\rightarrow$, $\Box$, and $\forall$. Formulas are defined as usual.

A *monadic modal predicate logic* is a set of monadic modal predicate formulas that includes the propositional logic $\mathbf{K}$ and the monadic classical predicate tautologies and is closed under Predicate Substitution, Modus Ponens, Generalisation, and Necessitation. The minimal such logic will be called[1] $\mathbf{QK}$. If $\mathbf{\Lambda}$ is a propositional 1-modal logic, then $\mathbf{Q\Lambda} := \mathbf{QK} + \mathbf{\Lambda}$ and $\mathbf{Q\Lambda C} := \mathbf{Q\Lambda} + Ba$, where $Ba := \forall x \Box P(x) \rightarrow \Box \forall x P(x)$ is the Barcan formula.

A *predicate Kripke frame* over a Kripke frame $F = (W, R)$ is a pair $\boldsymbol{F} = (F, D)$, where $D = (D_w)_{w \in W}$, with $D_w \neq \varnothing$ for all w and with $D_w \subseteq D_v$ whenever wRv.

A *valuation* on $\boldsymbol{F}$ is a family $\xi = (\xi_w)_{w \in W}$ of local valuations: $\xi_w(P_k^1) \subseteq D_w$ and $\xi_w(P_k^0) \in \{0, 1\}$. A *predicate Kripke model* over $\boldsymbol{F}$ is a pair $M = (\boldsymbol{F}, \xi)$, where ξ is a valuation on $\boldsymbol{F}$.

The truth relation $\Vdash$ between points w of a predicate Kripke model M and D_w-sentences (i.e., sentences obtained from formulas by replacing parameters with elements of D_w) is defined by recursion:

- $M, w \Vdash P_k^0$ if $\xi_w(P_k^0) = 1$;
- $M, w \Vdash P_k^1(a)$ if $a \in \xi_w(P_k^1)$;

[1] Usually, $\mathbf{QK}$, $\mathbf{Q\Lambda}$, and $\mathbf{Q\Lambda C}$ denote modal logics in languages with predicates of any arity, but in this paper we use the same notation for logics in languages with only monadic and nullary predicate letters.

- $M, w \Vdash \forall x\, A_1(x)$ if $M, w \Vdash A_1(a)$ whenever $a \in D_w$,

and the clauses for $\bot$, $\to$, $\Box$ are as in the propositional case.

A modal predicate formula A is *true in* M (in symbols, $M \models A$) if $M, u \Vdash \overline{\forall} A$ whenever $w \in W$. The formula A is *valid* on a predicate Kripke frame $\boldsymbol{F}$ (in symbols, $\boldsymbol{F} \models A$) if $M \models A$ whenever M is a Kripke model over $\boldsymbol{F}$. If L is a predicate modal logic, an *L-frame* is a predicate frame $\boldsymbol{F}$ validating all formulas from L; in this case, we write $\boldsymbol{F} \vDash L$.

By Soundness theorem [14, Theorem 3.2.29], $\mathbf{ML}(\boldsymbol{F}) := \{A \mid \boldsymbol{F} \models A\}$ is a modal predicate logic (called *the logic of* $\boldsymbol{F}$). The *modal predicate logic of a class* $\mathcal{C}$ *of predicate frames* (or the logic *determined by* $\mathcal{C}$) is logic defined as follows:

$$\mathbf{ML}(\mathcal{C}) \quad := \quad \bigcap \{\mathbf{ML}(\boldsymbol{F}) \mid \boldsymbol{F} \in \mathcal{C}\};$$

such logics are said to be *Kripke complete.* Every predicate logic L has the least Kripke complete extension, called the *Kripke completion*, which is the logic $\widehat{L}$ of the class of all L-frames.

3 1-variable predicate modal logics, semiproducts, and products

Let us recall definitions of some classes of monadic predicate modal formulas:

- *1-parametric* formulas contain at most one parameter;
- *1-variable* formulas are monadic containing at most one (fixed) variable x;
- *pure 1-variable* formulas are 1-variable without proposition letters;
- in *monodic* formulas [7, 3] every subformula of the form $\Box A$ is 1-parametric.

Monadic monodic fragments (mm-fragments) of logics **QK**, **QT**, **QK4**, and **QS4** are decidable [7, Theorem 5.1].[2] Even though they are syntactically more restrictive, 1-variable fragments are as expressive as mm-fragments:

Lemma 3.1.

(1) *Every mm-formula is* **QK***-equivalent to a Boolean combination of 1-variable formulas.*

[2] These are probably the largest known decidable fragments of modal predicate logics; most of 2-variable fragments, even in signatures with a single monadic predicate letter, are undecidable [8].

(2) *every 1-parametric mm-formula is* **QK***-equivalent to a 1-variable formula.*

Moreover, every 1-variable formula A in proposition letters $q_1, q_2, \dots, q_n$ translates into a pure 1-variable formula

$$A_0 \quad := \quad [\forall x\, Q_1(x), \dots, \forall x\, Q_n(x) / q_1, \dots, q_n]\, A,$$

where $Q_1, \dots, Q_n$ are monadic letters not occurring in A. Since, for every modal predicate logic,

$$L \vdash A \quad \Longleftrightarrow \quad L \vdash A_0,$$

we may assume that all 1-variable formulas are pure.

Furthermore, there exists a validity-preserving bijection $A \mapsto A_*$ between pure 1-variable modal predicate formulas and 2-modal propositional formulas:

$$\begin{array}{lcl} P_i(x)_* & := & p_i; \\ \bot_* & := & \bot; \\ (A \to B)_* & := & A_* \to B_*; \\ (\Box A)_* & := & \Box_1 A_*; \\ (\forall x\, A)_* & := & \Box_2 A_*. \end{array}$$

The *1-variable fragment* of a modal predicate logic L is the set

$$(L{-}1)^* \quad := \quad \{A \in L \mid A \text{ is a pure 1-variable formula}\}.$$

The *propositional counterpart* of $(L{-}1)^*$ is the set

$$L{-}1 \quad := \quad \{A_* \mid A \in L,\ A \text{ is a pure 1-variable formula}\}.$$

Loosely, we sometimes refer to the set $L{-}1$ as the 1-variable fragment of L.

Remark 3.2. The notion of Kripke completeness is also applicable to 1-variable fragments of predicate logics: $(L-1)^*$ is said to be Kripke complete if there exists a class $\mathcal{C}$ of predicate frames such that $(\mathbf{ML}(\mathcal{C})-1)^* = (L-1)^*$, or equivalently, $\mathbf{ML}(\mathcal{C}){-}1 = L{-}1$. Obviously, Kripke completeness of L implies Kripke completeness of $(L{-}1)^*$.

Lemma 3.3. *Let L be a modal predicate logic. Then,*

(1) *$L{-}1$ is a 2-modal propositional logic containing* $\mathbf{K} \sqcup \mathbf{S5}$.

(2) *If $L \vdash Ba$, then* $[\mathbf{K}, \mathbf{S5}] \subseteq L{-}1$.

Propositon 3.4. *Let* $\mathbf{\Lambda}$ *be a propositional 1-modal logic. Then,*

(1) $\mathbf{\Lambda} \lrcorner \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1 \subseteq \widehat{\mathbf{Q\Lambda}}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. *Hence, if* $\mathbf{Q\Lambda}$ *is Kripke complete, then*

$$\mathbf{\Lambda} \lrcorner \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}.$$

(2) $[\mathbf{\Lambda}, \mathbf{S5}] \subseteq \mathbf{Q\Lambda C}{-}1 \subseteq \widehat{\mathbf{Q\Lambda C}}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}$. *Hence, if* $\mathbf{Q\Lambda C}$ *is Kripke complete, then*

$$[\mathbf{\Lambda}, \mathbf{S5}] \subseteq \mathbf{Q\Lambda C}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}.$$

Definition 3.5. *A 1-modal propositional logic* $\mathbf{\Lambda}$ *is called* quantifier-friendly *if* $\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \lrcorner \mathbf{S5}$ *and* Barcan-friendly *if* $\mathbf{Q\Lambda C}{-}1 = [\mathbf{\Lambda}, \mathbf{S5}]$.

Proposition 3.4(1) implies that there exist four possibilities for semiproducts:

(1S) $\mathbf{\Lambda} \lrcorner \mathbf{S5} = \mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$,

(2S) $\mathbf{\Lambda} \lrcorner \mathbf{S5} = \mathbf{Q\Lambda}{-}1 \subset \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$,

(3S) $\mathbf{\Lambda} \lrcorner \mathbf{S5} \subset \mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.

(4S) $\mathbf{\Lambda} \lrcorner \mathbf{S5} \subset \mathbf{Q\Lambda}{-}1 \subset \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.

(1S) means that $\mathbf{\Lambda}$ and $\mathbf{S5}$ are semiproduct-matching. Some logics $\mathbf{\Lambda}$ of this type are described in Theorem 2.19. Another set of examples is presented in Section 6.

(2S) means that $\mathbf{\Lambda}$ and $\mathbf{S5}$ are not semiproduct-matching, but $\mathbf{\Lambda}$ is quantifier-friendly. Examples are given in Section 4.

Examples for (3S) are the logics $\mathbf{Alt}_n$, as shown in Section 4. Examples for (4S) are not known.

Proposition 3.4(2) implies that there exist four possibilities for products:

(1P) $[\mathbf{\Lambda}, \mathbf{S5}] = \mathbf{Q\Lambda C}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}$,

(2P) $[\mathbf{\Lambda}, \mathbf{S5}] = \mathbf{Q\Lambda C}{-}1 \subset \mathbf{\Lambda} \times \mathbf{S5}$,

(3P) $[\mathbf{\Lambda}, \mathbf{S5}] \subset \mathbf{Q\Lambda C}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}$.

(4P) $[\mathbf{\Lambda}, \mathbf{S5}] \subset \mathbf{Q\Lambda C}{-}1 \subset \mathbf{\Lambda} \times \mathbf{S5}$.

(1P) means that $\mathbf{\Lambda}$ and $\mathbf{S5}$ are product-matching. Examples are well known, see Theorem 2.19. Examples for (3P) are logics $\mathbf{Alt}_n$, as shown in Section 4.

(2P) means that $\mathbf{\Lambda}$ and $\mathbf{S5}$ are not product-matching, but $\mathbf{\Lambda}$ is Barcan-friendly; examples are unknown. Examples for (4P) are also unknown.

4 Logics not semiproduct-matching with S5

The following result was first stated, without a proof, in [9]:

Theorem 4.1. *Let* $\mathbf{\Lambda}$ *be a propositional 1-modal logic such that* $\Box \cdot \mathbf{T} \subseteq \mathbf{\Lambda} \subseteq$ $\mathbf{SL4}$*. Then,*

$$\mathbf{\Lambda} \leftthreetimes \mathbf{S5} \quad \subset \quad \mathbf{\Lambda} \leftthreetimes \mathbf{S5} + \Box_1 \Box_2 \mathit{ref}_1 \quad \subseteq \quad \mathbf{\Lambda} \rightthreetimes \mathbf{S5},$$

where $\mathit{ref}_1 = \Box_1 p \to p$*. Hence,* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are not semiproduct-matching.*

Proof. Let M_0 be the model from Fig. 1; then, $M_0, u \vDash \Diamond_1 \Diamond_2 \neg ref$ and $F_0 \vDash \mathbf{SL4} \leftthreetimes \mathbf{S5}$. Hence, $\Box_1 \Box_2 \mathit{ref}_1 \notin \mathbf{SL4} \leftthreetimes \mathbf{S5}$, which proves that the first inclusion is proper.

In view of Lemma 2.15 (3), to prove the second inclusion, it is enough to show that $\Box_1 \Box_2 \mathit{ref}_1 \in \Box \cdot \mathbf{T} \rightthreetimes \mathbf{S5}$. This membership follows from the validity of the formula $\Box_1 \Box_2 \mathit{ref}_1$ on every semiproduct of a $\Box \cdot \mathbf{T}$-frame with an $\mathbf{S5}$-frame. ■

Theorem 4.1 implies that the analogue of Theorem 2.19 does not hold for semiproducts (cf. Theorem 2.20); in particular, it gives us the following counterexamples:

Corollary 4.2. *Horn axiomatizable logics* $\Box \cdot \mathbf{T}$*,* $\mathbf{K5}$*, and* $\mathbf{K45}$ *are not semiproduct-matching with* $\mathbf{S5}$*.*

Moreover, the following nontrivial result is also true (stated in [10]; the proof is in preparation):

Theorem 4.3. *Every Kripke complete Horn axiomatizable logic is quantifier-friendly.*

Corollary 4.4. *The logics* $\Box \cdot \mathbf{T}$*,* $\mathbf{K5}$*, and* $\mathbf{K45}$ *satisfy (2S).*

Remark 4.5. A standard modal logic argument shows that there is a continuum of logics between $\Box \cdot \mathbf{T}$ and $\mathbf{SL4}$. Due to Theorem 4.1, this gives us a continuum of logics not satisfying (1S).

Remark 4.6. If $\mathbf{\Lambda}$ is a logic from the statement of Theorem 4.1, then $\mathbf{Q\Lambda}$ is Kripke incomplete [11, Theorem 5.11].

We next recall a well-known property of Jankov–Fine formulas X_G (this property is stated in [12] in a slightly different form):

Proposition 4.7. *Let G be a rooted n-transitive N-frame. Then there exists an N-modal formula X_G such that, for every n-transitive frame F, the following holds: $F \not\models X_G$ iff there exists a p-morphism from some cone of F onto G.*

Theorem 4.8. *If $\mathbf{Alt}_n \subseteq \mathbf{\Lambda} \subseteq \mathbf{Alt}_n + \Box^2\bot$, where $n \geqslant 3$, then $\mathbf{\Lambda}$ and $\mathbf{S5}$ are neither semiproduct- nor product-matching.*

Proof. We sketch the proof for $n = 4$; the general case is argued similarly. Let $G = (W, R, S)$ be the frame depicted in Fig. 2, on the right (S-reflexive accessibilities are not drawn).

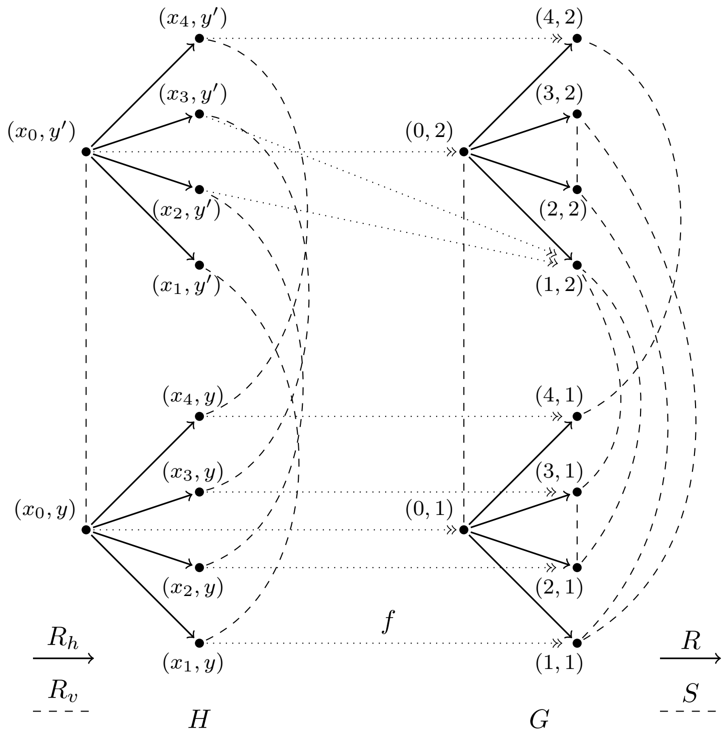

Figure 2. Frame G is not a p-morphic image of any $H \in \mathbf{V}(\mathbf{Alt}_4) \ltimes \mathcal{U}$

It is not hard to see that $G \models [\mathbf{Alt}_4 + \Box^2\bot, \mathbf{S5}]$. Hence, G is 3-transitive.[3] Let X_G be the Jankov–Fine formula of G, and let $A := \Box_1^2\bot \to X_G$. Surely, $G \models \Box_1^2\bot$, and, by Proposition 4.7, $G \not\models X_G$. Therefore, $G \not\models A$, and hence $A \notin [\mathbf{Alt}_n + \Box^2\bot, \mathbf{S5}]$.

On the other hand, $A \in \mathbf{Alt}_n \ltimes \mathbf{S5}$, since otherwise, by Proposition 4.7, G is a p-morphic image of a cone $F{\uparrow}(x_1, x_2)$, where F is a semiproduct of $F_1 \models \mathbf{Alt}_n$ and $F_2 \models \mathbf{S5}$. By Lemma 2.12, this cone is a semiproduct of $F{\uparrow}x_1$ and $F{\uparrow}x_2$.

[3]This fact can also be inferred from Fig. 2.

By Lemma 2.6, $F{\uparrow}x_1$ is an $\mathbf{Alt}_n$-frame. Since $F{\uparrow}x_2$ is a cone in an $\mathbf{S5}$-frame, it is a *cluster* (a frame with a universal relation).

However, as we next show, G cannot be a p-morphic of a semiproduct of such frames. Indeed, suppose f is a required p-morphism with $f(x_0, y) = (0, 1)$. By the lift property, there exist points (x_i, y), with $1 \leqslant i \leqslant 4$, and (x_0, y') such that $f(x_i, y) = (i, 1)$, for $i \in \{1, \ldots, 4\}$, and $f(x_0, y') = (0, 2)$ (see Fig. 2). Then, by monotonicity, $f(x_0, y')Rf(x_2, y')$ and $f(x_2, y)S(x_2, y')$; hence, $f(x_2, y') = (1, 2)$. Similarly, $f(x_3, y') = (1, 2)$ and $f(x_4, y') = (4, 2)$. Hence, (x_1, y') is mapped to either $(2, 2)$ or $(3, 2)$, which means that one of these points is not in the range of f, in contradiction with f being a p-morphism. ■

Recall that a modal predicate logic L is *strongly Kripke complete* if every L-consistent theory is satisfiable at a point of a model over an L-frame. By using selective submodels of canonical models (the method described in [11]), we can obtain the following result (for details, see [13]):

Theorem 4.9. *Every logic* $\mathbf{QAlt}_n$ *is strongly Kripke complete.*

Since adding closed propositional formulas preserves strong Kripke completeness, the following is also true:

Corollary 4.10. *Every logic* $\mathbf{QAlt}_n + \Box^m\bot$*, with* $m \geqslant 2$*, is strongly Kripke complete.*

The correspondent of the propositional formula alt_n is a classical first-order universal sentence. Hence, by Tanaka–Ono theorem [14, Theorem 7.4.7], we obtain the following:

Theorem 4.11. *If*

$$\mathbf{\Lambda} \in \{\mathbf{Alt}_n \mid n \geqslant 1\} \cup \{\mathbf{Alt}_n + \Box^m\bot \mid n \geqslant 1, m \geqslant 2\},$$

then the logic $\mathbf{Q\Lambda C}$ *is strongly Kripke complete.*

This implies the following:

Theorem 4.12. *If* $\mathbf{\Lambda} = \mathbf{Alt}_n$ *or* $\mathbf{\Lambda} = \mathbf{Alt}_n + \Box^m\bot$*, with* $n \geqslant 3$ *and* $m \geqslant 2$*, then the logic* $\mathbf{\Lambda} \lrcorner \mathbf{S5}$ *satisfies* (*3S*) *and the logic* $\mathbf{\Lambda} \times \mathbf{S5}$ *satisfies* (*3P*)*.*

Proof. By Theorem 4.8, $\mathbf{\Lambda} \leftthreetimes \mathbf{S5} \neq \mathbf{\Lambda} \lrcorner \mathbf{S5}$ and $[\mathbf{\Lambda}, \mathbf{S5}] \neq \mathbf{\Lambda} \times \mathbf{S5}$. By Theorem 4.9, Corollary 4.10, and Proposition 3.4(1), $\mathbf{Q\Lambda} - 1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. Hence, $\mathbf{\Lambda}_1 \lrcorner \mathbf{S5} \neq \mathbf{Q\Lambda}{-}1$. Similarly, it follows, by Theorem 4.11 and Proposition 3.4(2), that $\mathbf{Q\Lambda C} - 1 = \mathbf{\Lambda} \times \mathbf{S5}$. Hence, $\mathbf{Q\Lambda C} - 1 \neq [\mathbf{\Lambda}, \mathbf{S5}]$. ■

Problem 4.13. *Suppose that* $\mathbf{\Lambda}$ *is a logic from Theorem 4.11. Axiomatize logics* $\mathbf{Q\Lambda} - 1$ $(= \mathbf{\Lambda} \leftthreetimes \mathbf{S5})$ *and* $\mathbf{Q\Lambda C} - 1$ $(= \mathbf{\Lambda} \times \mathbf{S5})$*.*

5 Local tabularity and modal depth

We now recall definitions and facts from [15] about N-modal formulas and logics.

Definition 5.1. *The* modal depth $md(A)$ *of an N-modal propositional formula A is the maximal number of nested occurrences of modal operators in A:*

$$\begin{array}{lcl} md(\bot) & := & 0; \\ md(p_j) & := & 0; \\ md(A \to B) & := & \max(md(A), md(B)); \\ md(\Box_i A) & := & md(A) + 1. \end{array}$$

Definition 5.2. *The* modal depth $md_{\mathbf{\Lambda}}(A)$ of a formula A in an N-modal logic $\mathbf{\Lambda}$ *is defined as follows:*

$$md_{\mathbf{\Lambda}}(A) := \min\{md(B) \mid \mathbf{\Lambda} \vdash A \leftrightarrow B\}.$$

The modal depth $md(\mathbf{\Lambda})$ of a logic $\mathbf{\Lambda}$ *is defined as follows:*

$$md(\mathbf{\Lambda}) := \begin{cases} \max\{md_{\mathbf{\Lambda}}(B) \mid B \textit{ is an } N\textit{-modal propositional formula}\} & \textit{if exists;} \\ \infty & \textit{otherwise.} \end{cases}$$

Definition 5.3. *An N-modal logic $\mathbf{\Lambda}$ is* locally tabular *if, for any finite k, there exist only finitely many N-modal k-formulas non-equivalent in $\mathbf{\Lambda}$.*

Propositon 5.4.

(1) *Every locally tabular logic has the fmp.*

(2) *Every propositional modal logic of finite modal depth is locally tabular.*

6 Logics semiproduct-matching with S5

In this section, we show that each logic $\mathbf{K05} + \Box^n\bot$ is semiproduct-matching with **S5** and that the corresponding semiproduct has the semiproduct fmp. We use the following nomenclature for logics:

$$\begin{array}{lcl} \mathbf{\Lambda}_{0n} & := & \mathbf{K05} + \Box^n\bot; \\ \mathbf{\Lambda}_n & := & \mathbf{\Lambda}_{0n} \leftthreetimes \mathbf{S5}; \\ \mathbf{\Lambda}'_n & := & \mathbf{\Lambda}_n + rcom = [\mathbf{\Lambda}_{0n}, \mathbf{S5}]. \end{array}$$

Theorem 6.1. *If $n \geqslant 1$, then $md(\mathbf{\Lambda}_n) \leqslant 2n - 1$.*

The proof uses bisimulation games; for more details, see [15, 16]. Thus, by Proposition 5.4, we obtain the following:

Corollary 6.2. *The logics $\mathbf{\Lambda}_n$ and $\mathbf{\Lambda}'_n$ have the fmp.*

To prove semiproduct-matching and the semiproduct fmp for $\mathbf{\Lambda}_n$, it suffices to construct p-morphisms from semiproducts of finite $\mathbf{\Lambda}_{0n}$-frames with clusters onto finite $\mathbf{\Lambda}_n$-cones. Similarly, to prove product-matching and the product fmp for $\mathbf{\Lambda}'_n$, it suffices to construct p-morphisms from products of finite $\mathbf{\Lambda}_{0n}$-frames with clusters onto finite $\mathbf{\Lambda}'_n$-cones. We construct the sought p-morphisms in a number of steps (Lemmas 6.6 – 6.10).

Definition 6.3. *Let $F = (W, R_1, \ldots, R_N)$ and $F' = (W', R'_1, \ldots, R'_N)$ be frames. A map $g\colon W \longrightarrow W'$ is a* strong homomorphism *from F to F' if, for every $w, v \in W$ and every i,*

$$wR_iv \iff g(w)R'_ig(v).$$

Lemma 6.4. *Every surjective strong homomorphism is a p-morphism and an elementary equivalence for formulas without equality.*

Thus, if $\mathbf{\Lambda}$ is elementary (with respect to a classical signature without equality), then the class $\mathbf{V}(\mathbf{\Lambda})$ is closed under strong homomorphic pre-images.

Definition 6.5. *Let $F = (W, R_1, R_2)$ be a $\mathbf{K}\,\lrcorner\,\mathbf{S5}$-frame.*

- *A* row *in F is an equivalence class under the relation $(R_1 \cup R_1^{-1})^*$.*
- *A* column *in F is an equivalence class under R_2.*
- *A* block *in F is a non-empty intersection of a row and a column.*
- *F is* organized *if, for every row U in F, the frame $(W, R_1) \restriction U$ is rooted.*
- *F is* equalized *if every column in F consists of blocks of the same cardinality.*
- *F is* straight *if all its blocks are singletons.*

Lemma 6.6 (on organizing). *Every finite rooted $\mathbf{\Lambda}_n$-frame is a strong homomorphic image of a finite rooted organized $\mathbf{\Lambda}_n$-frame; a similar fact holds for $\mathbf{\Lambda}'_n$-frames.*

Proof. Let $F = (W, R_1, R_2)$ be a finite rooted $\mathbf{\Lambda}_n$-frame. We say that a point $a \in W$ is R_1-*minimal* if $R_1^{-1}(a) = \varnothing$. We put

$$V := \{(a, x) \mid a \text{ is } R_1\text{-minimal and } aR_1^*x\}$$

and define relations S_i on V:

$$(a,x)S_1(b,y) \iff a = b \ \& \ xR_1y,$$
$$(a,x)S_2(b,y) \iff xR_2y.$$

Then (V, S_1, S_2) is rooted and organized, and the map $(a,x) \mapsto x$ is a required strong homomorphism onto F. ■

Lemma 6.7 (on equalizing). *Every finite rooted organized $\mathbf{\Lambda}_n$-frame is a strong homomorphic image of a finite rooted equalized $\mathbf{\Lambda}_n$-frame; a similar fact holds for $\mathbf{\Lambda}'_n$-frames.*

Proof. To equalize a frame $F = (W, R_1, R_2)$, we add extra points to blocks making the blocks of a column the same size. We use the fact that, for every pair of blocks β and γ in F and each $k \in \{1,2\}$,

$$\exists x \in \beta \, \exists y \in \gamma \, xR_ky \iff \forall x \in \beta \, \forall y \in \gamma \, xR_ky. \tag{1}$$

Hence, we write $\beta R_k \gamma$ whenever there exist $x \in \beta$ and $y \in \gamma$ such that xR_ky. We replace each block β in F with a block β' whose cardinality is the largest for blocks in the column of β. We put $W' := \{\beta' \mid \beta \text{ is a block in } F\}$ and define, for every $x \in \beta'$, $y \in \gamma'$, and $k \in \{1,2\}$,

$$xR'_ky \iff \beta R_k \gamma.$$

Due to (1), each relation R'_k is well defined. Then the frame $F' := (W', R'_1, R'_2)$ is equalized. A surjective map sending each point of block β' in F' to some point of block β in F is a strong homomorphism. ■

Lemma 6.8 (on straightening). *Every finite rooted equalized $\mathbf{\Lambda}_n$-frame is a strong homomorphic image of a finite rooted straight $\mathbf{\Lambda}_n$-frame; a similar fact holds for $\mathbf{\Lambda}'_n$-frames.*

Proof. To straighten a frame $F = (W, R_1, R_2)$, we first construct a frame whose columns all have the size, say n, of the largest column in F. To that end, we put $W' = W \times n$ and define $R'_1 \subseteq W' \times W'$ so that

$$(x,i)R'_1(y,j) \iff xR_1y \ \& \ i = j,$$

and $R'_2 \subseteq W' \times W'$ so that, if β and γ are blocks from the same column of F, $x \in \beta$, and $y \in \gamma$, then, for fixed enumerations N_β of β and N_γ of γ,

$$(x,i)R'_2(y,j) \iff N_\beta(x) + i \equiv N_\gamma(y) + j \ (mod\ |\beta|).$$

Then the map $(x,i) \mapsto x$ is a strong homomorphism from $F' = (W', R'_1, R'_2)$ onto F. ■

From Lemmas 6.6–6.8 and 6.4 we immediately obtain the following:

Lemma 6.9. *Every finite rooted $\mathbf{\Lambda}_n$-frame is a p-morphic image of a finite rooted straight $\mathbf{\Lambda}_n$-frame; a similar fact holds for $\mathbf{\Lambda}'_n$-frames.*

Lemma 6.10. *Every finite rooted straight $\mathbf{\Lambda}_n$-frame is isomorphic to a semiproduct of an $\mathbf{\Lambda}_{0n}$-frame and a cluster; a similar fact holds for $\mathbf{\Lambda}'_n$-frames and products.*

Proof. If $F = (W, R_1, R_2)$ is a finite straight frame rooted at x_0, then F is isomorphic to a (semi)product of the frame $(R_1^*(x_0), R_1 \restriction R_1^*(x_0))$ and the cluster whose points are the rows of F. ■

Theorem 6.11.

(1) *The logics* $\mathbf{K05}+\Box^n\bot$ *and* $\mathbf{S5}$ *are both semiproduct-matching and product-matching.*

(2) *The logics* $(\mathbf{K05} + \Box^n\bot) \leftthreetimes \mathbf{S5}$ *have the semiproduct fmp, and the logics* $(\mathbf{K05} + \Box^n\bot) \times \mathbf{S5}$ *have the product fmp.*

Proof. Let, as before, $\mathbf{\Lambda}_{0n} := \mathbf{K05} + \Box^n\bot$, $\mathbf{\Lambda}_n := \mathbf{\Lambda}_{0n} \lrcorner \mathbf{S5}$, and $\mathbf{\Lambda}_n := [\mathbf{\Lambda}_{0n}, \mathbf{S5}]$.

(1) Suppose $A \notin \mathbf{\Lambda}_n$. By Corollary 6.2, $\mathbf{\Lambda}_n$ has the fmp. Hence, A is refuted on a finite rooted $\mathbf{\Lambda}_n$-frame. By Lemmas 6.9 and 6.10, this frame is a p-morphic image of a semiproduct of a finite $\mathbf{\Lambda}_{0n}$-frame and a finite cluster. Since p-morphisms preserve validity of modal formulas, $A \notin \mathbf{\Lambda}_{0n} \leftthreetimes \mathbf{S5}$. Thus, $\mathbf{\Lambda}_{0n} \leftthreetimes \mathbf{S5} \subseteq \mathbf{\Lambda}_n$. The converse is given by Lemma 2.15(3).

The proof for $\mathbf{\Lambda}'_n$ is similar.

(2) Since the (semi)product frames obtained in the proof of (1) are finite, the claim follows. ■

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