Undecidability of predicate modal and superintuitionistic logics with a single monadic letter and two variables

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Motivation

- Classical decision problem (David Hilbert): find an algorithm deciding validity in the classical predicate logic **QCL**.
- Solution: (Alonzo Church 1936, Alan Turing 1937): **QCL** is undecidable.
- Classical decision problem as a classification problem: identify the "maximal" decidable and the "minimal" undecidable fragments of **QCL**; a comprehensive overview can be found in the book [Börger, Grädel & Gurevich].
- Criteria:
  - the quantifier prefix: $\exists^*\forall^*$ decidable, $\forall^3\exists^*$ undecidable;
  - the number of variables: 2 decidable, 3 undecidable;
  - the number and arity of predicate letters: any number of monadic decidable, a single binary undecidable;
  - variables *and* predicate letters: the fragment with three variables and a single binary letter is undecidable [Tarski & Givant].

Motivation

- Non-classical decision problem as a classification problem: identify the “maximal” decidable and the “minimal” undecidable fragments of FO modal and superintuitionistic logics.
- S. Kripke 1962 Every modal logic validated by **S5** frames is undecidable with two monadic predicate letters: write $\Diamond(P_1(x) \wedge P_2(y))$ for $R(x, y)$ to obtain an embedding of an undecidable fragment of **QCL** (“Kripke trick”).

  NB This result can be strengthened to one monadic letter [D. Gabbay]:
  - $R(x, y) \;\mapsto\; \neg\Diamond(P(x) \wedge P(y))$, for a sib-relation R.
- S. Maslov, G. Mints, and V. Orevkov 1965 The intuitionistic predicate logic **QH** is undecidable with a single monadic predicate letter.
- Single-variable fragments are, as a rule, decidable (K. Segerberg, G. Fisher-Servi, H. Ono, G. Mints).

Motivation

- D. Gabbay and V. Shehtman 1993 Most natural predicate modal and superintuitionistic logics with the constant domain axiom are undecidable in languages with two individual variables.
- F. Wolter and M. Zakharyaschev 2001 Monodic fragments are decidable (a monodic fragment = a decidable fragment of **QCL** + applying modalities to formulas with at most one parameter).
- R. Kontchakov, A. Kurucz, and M. Zakharyaschev 2005
    - **QH** is undecidable with two variables, two binary predicate letters and an unrestricted supply of monadic letters;
    - most modal logics are undecidable with two variables and an unrestricted supply of monadic letters.
    - open problem #1: is the two-variable monadic fragment of **QH** decidable?
    - open problem #2: how many monadic predicates are needed for undecidability of two-variable modal logics?
- The current work addresses problems #1 and #2.

This talk

We show the following:

- sublogics of **QGL**, **QGrz**, and **QKTB** are undecidable with two variables and a single monadic predicate letter;
- superintuitionistic logics between **QH** and **QKC** (= **QH** + the weak excluded middle) are undecidable with two variables and a single monadic predicate letter.

Language

Intuitionistic formulas (= classical formulas):

$$\varphi \quad := \quad P(x_1, \ldots, x_n) \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \forall x \varphi \mid \exists x \varphi$$

Modal formulas:

$$\varphi \quad := \quad P(x_1, \ldots, x_n) \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \forall x \varphi \mid \exists x \varphi \mid \Box \varphi$$

NB No function symbols, constants, or equality!

Standard abbreviations:

$$\begin{array}{lcl} \neg\varphi & = & \varphi \to \bot; \\ \varphi \leftrightarrow \psi & = & (\varphi \to \psi) \wedge (\psi \to \varphi); \\ \Diamond\varphi & = & \neg\Box\neg\varphi. \end{array}$$

Logics

A first-order (classical normal) modal logic is a set of formulas including **QCL** and **K** and closed under (MP), (Sub), (Gen), and Necessitation.

A first-order superintuitionistic logic is a set of formulas including **QH** and closed under (MP), (Sub), and (Gen).

The minimal predicate extension of a propositional logic L (modal or superintuitionistic, depending on the context) is denoted by $\mathbf{Q}L$.

Expanding domains Kripke semantics: modal logics

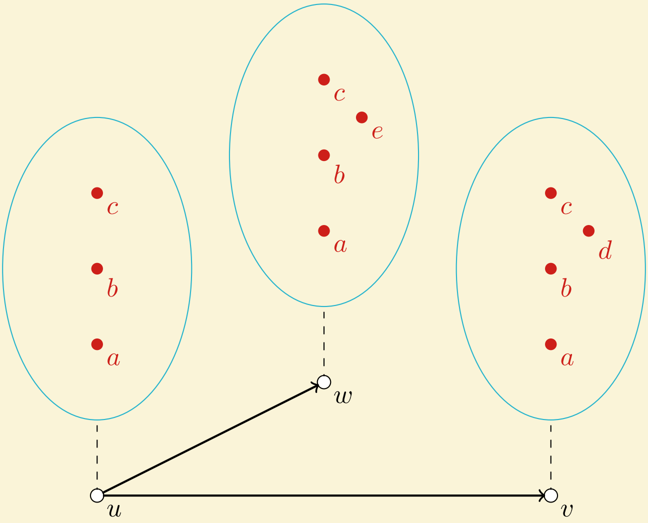
A Kripke frame is a pair $\mathfrak{F} = \langle W, R\rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$.

An augmented frame is a tuple $\boldsymbol{F} = \langle \mathfrak{F}, \Delta, D\rangle$, where $\mathfrak{F}$ is a Kripke frame, $\Delta \neq \varnothing$, and $D\colon W \to 2^{\Delta} \setminus \varnothing$ is a map satisfying the expanding domains condition

$$(*) \quad wRw' \implies D_w \subseteq D_{w'}.$$

A Kripke model is a pair $\langle \boldsymbol{F}, I\rangle$, where $\boldsymbol{F}$ is an augmented frame and $I(w, P) \subseteq D_w^n$ whenever $w \in W$ and P is an n-ary predicate letter (i.e., for every $w \in W$, the pair $\mathfrak{M}_w = \langle D_w, I_w\rangle$ is a classical model).

Augmented frames: an example



Expanding domains Kripke semantics: modal logics

An assignment is a map $g\colon Var \to \Delta$.

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^w$;
- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi \wedge \psi$ if $\mathfrak{M}, w \models^g \varphi$ and $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \vee \psi$ if $\mathfrak{M}, w \models^g \varphi$ or $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \to \psi$ if $\mathfrak{M}, w \not\models^g \varphi$ or $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \exists x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$, for some g' with $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$ whenever $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \Box\varphi$ if $\mathfrak{M}, w' \models^g \varphi$ whenever $w' \in R(w)$.

- $\mathfrak{M}, w \models \varphi$ if $\mathfrak{M}, w \models^g \bar{\forall}\varphi$, for some assignment g;
- $\mathfrak{M} \models \varphi$ if $\mathfrak{M}, w \models \varphi$, for every $w \in W$;
- $\boldsymbol{F} \models \varphi$ if $\mathfrak{M} \models \varphi$, for every model $\mathfrak{M}$ over $\boldsymbol{F}$;
- $\mathfrak{F} \models \varphi$ if $\boldsymbol{F} \models \varphi$, for every $\boldsymbol{F}$ over $\mathfrak{F}$.

$\mathbb{N} \times \mathbb{N}$ tiling

A tile is a square with coloured edges.

A tile type t is a quadruple of colours $\langle \mathit{left}(t), \mathit{up}(t), \mathit{right}(t), \mathit{down}(t) \rangle$.

Tiling problem: Given a finite set T of tile types, can we tile $\mathbb{N} \times \mathbb{N}$ with T-tiles so that the adjacent colours match? I. e., does there exist a function $f : \mathbb{N} \times \mathbb{N} \to T$ such that, for every $n, m \in \mathbb{N}$,

(T_1) $\mathit{right}(f(n, m)) = \mathit{left}(f(n + 1, m))$;

(T_2) $\mathit{up}(f(n, m)) = \mathit{down}(f(n, m + 1))$.

If such a function exists, we say that T *tiles* $\mathbb{N} \times \mathbb{N}$.

Reduction of tiling to modal satisfiability (Kontchakov, Kurucz & Zakharyaschev)

$$
\begin{array}{ll}
(1) & \forall x \bigvee_{t\in T}(P_t(x) \wedge \bigwedge_{t'\neq t} \neg P_{t'}(x));\\
(2) & \forall x \forall y\, (H(x,y) \to \bigwedge_{right(t)\neq left(t')} \neg(P_t(x) \wedge P_{t'}(y)));\\
(3) & \forall x \forall y\, (V(x,y) \to \bigwedge_{up(t)\neq down(t')} \neg(P_t(x) \wedge P_{t'}(y)));\\
(4) & \forall x \exists y\, H(x,y) \wedge \forall x \exists y\, V(x,y);\\
(5) & \forall x \forall y\, (H(x,y) \to \Box H(x,y));\\
(6) & \forall x \forall y\, (V(x,y) \to \Box V(x,y));\\
(7) & \forall x \forall y\, (\Diamond V(x,y) \to V(x,y));\\
(8) & \forall x\, \Diamond D(x);\\
(9) & \Box \forall x \forall y\, [V(x,y) \wedge \exists x\, (D(x) \wedge H(y,x)) \to \\
 & \qquad \forall y (H(x,y) \to \forall x (D(x) \to V(y,x)))],
\end{array}
$$

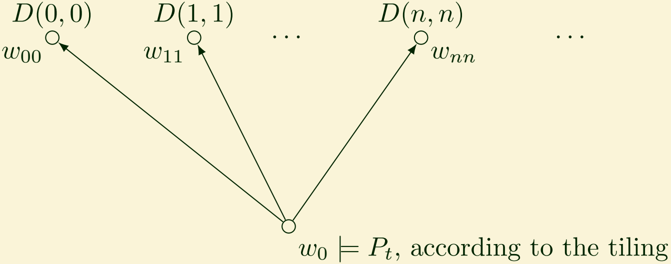
Reduction of tiling to 2-variable modal formulas (Kontchakov, Kurucz & Zakharyaschev)

Theorem (Kontchakov, Kurucz & Zakharyaschev)

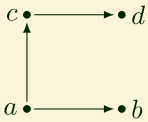
Let L be propositional modal logic valid on a Kripke frame with a world that sees infinitely many worlds and let χ_T be the conjunction of formulas (1) through (9). Then χ_T is $\mathbf{Q}L$-satisfiable iff T tiles $\mathbb{N} \times \mathbb{N}$.

('if') For every $w \in W$,

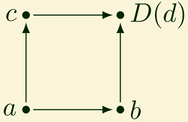
- $D(w) = \mathbb{N} \times \mathbb{N}$,
- $H^w(\langle i, j\rangle, \langle i+1, j\rangle)$ and $V^w(\langle i, j\rangle, \langle i, j+1\rangle)$.



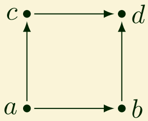
('only if') Assume that $\mathfrak{M}, w \models \chi_T$. Suppose we have the following at w:



Then, by (8), there exists $w_d \in R(w)$ such that $w_d \models D(d)$. Hence, by (9), we have the following at w_d:

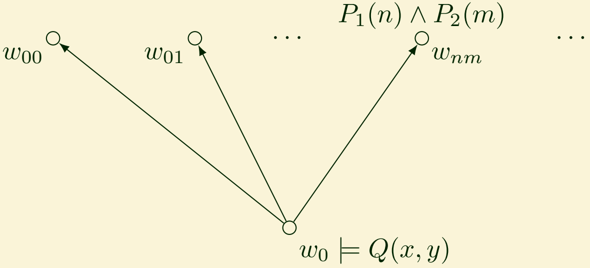


Hence, as needed, by (7), at w,



Kripke trick

Substitution into a classical formula: $Q(x,y) \mapsto \Diamond(P_1(x) \wedge P_2(y))$.



Reduction of tiling to monadic 2-variable modal formulas (a variation on KKZ)

$$(1) \quad \forall x \bigvee_{t \in T} (P_t(x) \wedge \bigwedge_{t' \neq t} \neg P_{t'}(x));$$

$$(2') \quad \forall x \forall y \, (\Diamond(H_1(x) \wedge H_2(y)) \to \bigwedge_{right(t) \neq left(t')} \neg(P_t(x) \wedge P_{t'}(y)));$$

$$(3') \quad \forall x \forall y \, (\Diamond(V_1(x) \wedge V_2(y)) \to \bigwedge_{up(t) \neq down(t')} \neg(P_t(x) \wedge P_{t'}(y)));$$

$$(4') \quad \forall x \exists y \, \Diamond(H_1(x) \wedge H_2(y)) \wedge \forall x \exists y \, \Diamond(V_1(x) \wedge V_2(y));$$

$$(5') \quad \forall x \forall y \, (\Diamond(H_1(x) \wedge H_2(y)) \to \Box(\forall x Q(x) \to \Diamond(H_1(x) \wedge H_2(y))));$$

$$(6') \quad \forall x \forall y \, (\Diamond(V_1(x) \wedge V_2(y)) \to \Box(\forall x Q(x) \to \Diamond(V_1(x) \wedge V_2(y))));$$

$$(7') \quad \forall x \forall y \, (\Diamond(\forall x Q(x) \wedge \Diamond(V_1(x) \wedge V_2(y))) \to \Diamond(V_1(x) \wedge V_2(y)));$$

$$(8') \quad \forall x \, \Diamond(\forall x Q(x) \wedge D(x));$$

$$(9') \quad \Box(\forall x Q(x) \to \forall x \forall y \, [\Diamond(V_1(x) \wedge V_2(y)) \wedge \exists x \, (D(x) \wedge \Diamond(H_1(y) \wedge H_2(x))) \to$$
$$\forall y (\Diamond(H_1(x) \wedge H_2(y)) \to \forall x (D(x) \to \Diamond(V_1(y) \wedge V_2(x))))]),$$

We want the following add-ons to the previous Theorem:

- we want a monadic formula (i.e., no binary letters);
- we want not just a $\mathbf{Q}L$-satisfiable formula, but a formula with the following property:

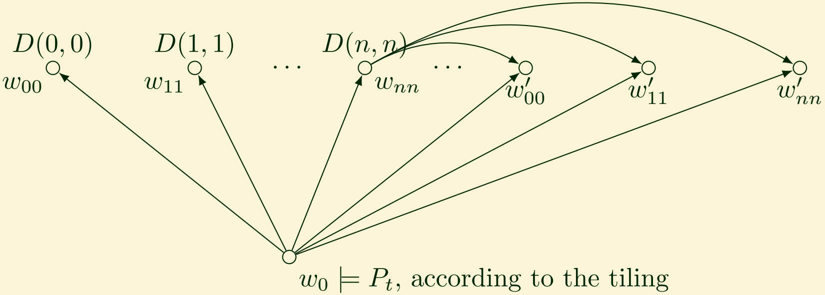
Definition

We say that a monadic formula φ is $\mathbf{Q}L$*-suitable* if φ is satisfiable in a model $\mathfrak{M} \models \mathbf{Q}L$ with the downward heredity property:
$\mathfrak{M} \models \Diamond P(a) \to P(a)$, for every monadic letter P and every $a \in D(w)$.

Theorem

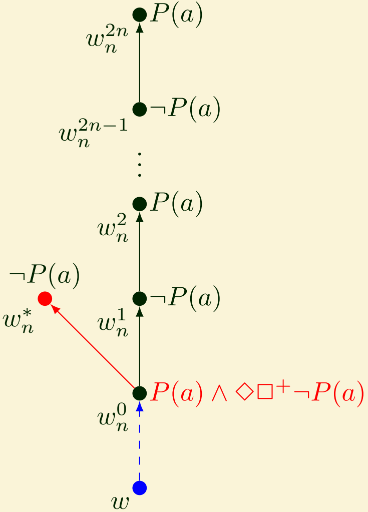
Let L be propositional modal logic valid on a Kripke frame with a world w_0 and two infinite disjoint sets of worlds U and U' such that $w_0 R w$ whenever $w \in U \cup U'$ and uRu' whenever $u \in U$ and $u' \in U'$, and let φ_T be the conjunction of formulas $(1')$ through $(9')$. Then φ_T is $\mathbf{Q}L$-suitable iff T tiles $\mathbb{N} \times \mathbb{N}$.

('if')



('only if') Pretty much as in the previous Theorem.

Simulation of monadic predicate letters with P only



$$A_k(x) \quad = \quad P(x) \wedge \Diamond\Box^+\neg P(a) \wedge (\neg P(x) \text{ and } P(x) \text{ alternate } n \text{ times}).$$

Then $B_k = \Diamond A_k(x)$ simulates $P_k(x)$ at w.

Simulation of monadic predicate letters with P only

We use $B_k(x)$ to simulate a monadic predicate $P_k(x)$.

Before substituting $P_k(x) \mapsto B_k(x)$, we need to relativize φ_T (assume that the monadic letters of φ are $P_1, \dots, P_s$) to $\forall x\, P_{s+1}(x) \wedge \varphi_T^*$, where the translation $(\cdot)^*$ recursively replaces $\Box\psi$ with $\forall x\, P_{s+1}(x) \to \psi^*$.

Finally, we substitute $P_k(x) \mapsto B_k(x)$ into $\forall x\, P_{s+1}(x) \wedge \varphi_T^*$.

This works for sublogics of **QGL** and **QGrz**.

NB Transitivity is taken care of since we work with downward hereditary models: this is why we wanted L-suitability rather than L-satisfiability.

With a bit of fiddling, a similar construction can be done for sublogics of **QKTB**.

Single monadic letter: results

Theorem

Every sublogic of **QGL** *and every sublogic of* **QGrz** *is undecidable (more precisely, Σ^0_1-complete) with two individual variables and a single monadic predicate letter.*

Theorem

Every sublogic of **QKTB** *is undecidable (more precisely, Σ^0_1-complete) with two individual variables and a single monadic predicate letter.*

Since we worked throughout with augmented frames with locally constant domains ($wRw' \Longrightarrow D(w) = D(w')$), we also obtain the following:

Corollary

Everything works for logics with the Barcan formula.

Open problem

Problem

What about **QS5**?

Conjecture

QS5 *with two variables and a single monadic predicate letter is decidable.*

Kripke semantics for si logics

Kripke frames are posets. Augmented frames are defined as in modal Kripke semantics with expanding domains.

A Kripke model is a pair $\langle \boldsymbol{F}, I \rangle$, where $\boldsymbol{F}$ is an augmented frame and $I(w, P) \subseteq D_w^n$ whenever $w \in W$ and P is an n-ary predicate letter (i.e., for every $w \in W$, the pair $\mathfrak{M}_w = \langle D_w, I_w \rangle$ is a classical model) satisfying the heredity condition:

$$wRw' \implies I(w, P) \subseteq I(w', P).$$

Kripke semantics for si logics

- $\mathfrak{M}, w \Vdash^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^w$;
- $\mathfrak{M}, w \nVdash^g \bot$;
- $\mathfrak{M}, w \Vdash^g \varphi \wedge \psi$ if $\mathfrak{M}, w \Vdash^g \varphi$ and $\mathfrak{M}, w \Vdash^g \psi$;
- $\mathfrak{M}, w \Vdash^g \varphi \vee \psi$ if $\mathfrak{M}, w \Vdash^g \varphi$ or $\mathfrak{M}, w \Vdash^g \psi$;
- $\mathfrak{M}, w \Vdash^g \varphi \to \psi$ if $\mathfrak{M}, w' \nVdash^g \varphi$ or $\mathfrak{M}, w' \Vdash^g \psi$ whenever $w' \in R(w)$;
- $\mathfrak{M}, w \Vdash^g \exists x\, \varphi$ if $\mathfrak{M}, w \Vdash^{g'} \varphi$, for some g' with $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \Vdash^g \forall x\, \varphi$ if $\mathfrak{M}, w' \Vdash^{g'} \varphi$ whenever $w' \in R(w)$,
  $g' \stackrel{x}{=} g$ and $g'(x) \in D_{w'}$.

Truth and validity are defined analogously to the modal Kripke semantics.

Reduction of tiling to refutable *positive* 2-variable formulas

$$\forall x \bigvee_{t\in T}(P_t(x) \wedge \bigwedge_{t'\neq t}(P_{t'}(x)\to q)), \quad (1)$$

$$\bigwedge_{right(t)\neq left(t')} \forall x\,\forall y\,(H(x,y) \wedge P_t(x) \wedge P_{t'}(y)\to q), \quad (2)$$

$$\bigwedge_{up(t)\neq down(t')} \forall x\,\forall y\,(V(x,y) \wedge P_t(x) \wedge P_{t'}(y)\to q), \quad (3)$$

$$\forall x\,\exists y\,H(x,y) \wedge \forall x\,\exists y\,V(x,y), \quad (4)$$

$$\forall x\,\forall y\,(V(x,y) \vee (V(x,y)\to q)), \quad (5)$$

$$\forall x\,\forall y\,[V(x,y)\wedge\exists x\,(D(x)\wedge H(y,x)) \to \forall y\,(H(x,y) \to \forall x\,(D(x) \to V(y,x)))]. \quad (6)$$

We want positive formulas, so we use $\varphi \to q$ instead of $\varphi \to \bot$. The use of $\bot$ would cause problems with latter stages of the reduction.

Let ψ_T be the conjunction of formulas (1) through (6), and

$$\varphi_T = \psi_T \to (\exists x\, (D(x) \to q)).$$

Theorem (Kontchakov, Kurucz & Zakharyaschev)

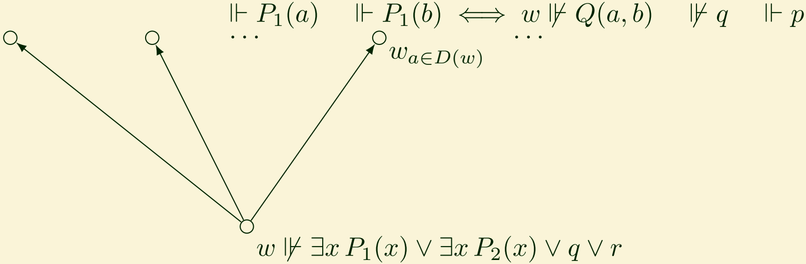
*$\varphi_T \notin$ **QH** iff T tiles $\mathbb{N} \times \mathbb{N}$.*

Corollary

The positive *fragment of* **QH** *with two variables and only binary and monadic predicate letters is undecidable.*

Kripke trick for intuitionistic formulas

Let q and p be nullary predicate letters and let the formula $\bar{\varphi}$ be obtained from φ by substitution $Q(x,y) \mapsto (P_1(x) \wedge P_2(y) \to q) \vee p$.



Kripke trick for intuitionistic formulas

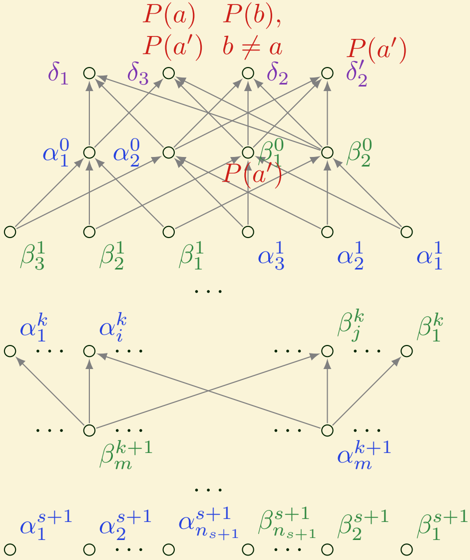
Theorem

Let φ be a positive formula containing no predicate letters other than Q and let $\mathbf{QH} \subseteq L \subseteq \mathbf{QKC}$. Then $\varphi \in L$ iff $\bar{\varphi} \in L$.

Corollary

The positive *fragment of* $\mathbf{QH}$ *with two variables and only monadic predicate letters is undecidable.*

Simulation of monadic predicate letters with P only



Simulation of monadic predicate letters with P only

First, we define formulas associated with the worlds of the three top-most levels:

$$
\begin{array}{rcl}
D_1 & = & \exists x\, P(x); \\
D_2(x) & = & \exists x\, P(x) \to P(x); \\
D_3(x) & = & P(x) \to \forall x\, P(x); \\
\\
A_1^0(x) & = & D_2(x) \to D_1 \vee D_3(x); \\
A_2^0(x) & = & D_3(x) \to D_1 \vee D_2(x); \\
B_1^0(x) & = & D_1 \to D_2(x) \vee D_3(x); \\
B_2^0(x) & = & A_1^0(x) \wedge A_2^0(x) \wedge B_1^0(x) \to D_1 \vee D_2(x) \vee D_3(x); \\
\\
A_1^1(x) & = & A_1^0(x) \wedge A_2^0(x) \to B_1^0(x) \vee B_2^0(x); \\
A_2^1(x) & = & A_1^0(x) \wedge B_1^0(x) \to A_2^0(x) \vee B_2^0(x); \\
A_3^1(x) & = & A_1^0(x) \wedge B_2^0(x) \to A_2^0(x) \vee B_1^0(x); \\
B_1^1(x) & = & A_2^0(x) \wedge B_1^0(x) \to A_1^0(x) \vee B_2^0(x); \\
B_2^1(x) & = & A_2^0(x) \wedge B_2^0(x) \to A_1^0(x) \vee B_1^0(x); \\
B_3^1(x) & = & B_1^0(x) \wedge B_2^0(x) \to A_1^0(x) \vee A_2^0(x).
\end{array}
$$

Simulation of monadic predicate letters with P only

We proceed by recursion. Assume formulas associated with the worlds of level k, where $k \geqslant 1$, have been defined. Let i, j and m be as in the definition of frame $\mathfrak{F}_0$ above; put

$$\begin{array}{rcl} A_m^{k+1}(x) & = & A_1^k(x) \to B_1^k(x) \vee A_i^k(x) \vee B_j^k(x); \\ B_m^{k+1}(x) & = & B_1^k(x) \to A_1^k(x) \vee A_i^k(x) \vee B_j^k(x). \end{array}$$

Simulation of monadic predicate letters with P only

Lemma

Let $\mathfrak{N}_a$ be an a-suitable model with a constant domain $\mathcal{A}$. Then,

$$\begin{array}{lcl} \mathfrak{N}_a, w \not\models A^k_m(a) & \iff & wR_0\alpha^k_m; \\ \mathfrak{N}_a, w \not\models B^k_m(a) & \iff & wR_0\beta^k_m. \end{array}$$

Lemma

Let $\mathfrak{N}_a$ be an a-suitable model with a constant domain $\mathcal{A}$ and let $b \in \mathcal{A} - \{a\}$. Then, for every $w \in W_0$ and every $k \geqslant 2$,

$$\mathfrak{N}_a, w \models A^k_m(b) \quad \text{and} \quad \mathfrak{N}_a, w \models B^k_m(b).$$

Simulation of monadic predicate letters with P only

Suppose φ contains letters $P_1, \dots, P_s$. Let $\varphi^{\#}$ be the result of the following substitution into $\bar{\varphi}$ (the formula obtained at the previous stage of reduction), for each $r \in \{1, \dots, s\}$,

$$P_r(x) \mapsto A_r^{s+1}(x) \vee B_r^{s+1}(x).$$

Lemma

Let $L \in [\mathbf{QH}, \mathbf{QKC}]$. Then, $\varphi \in L$ iff $\varphi^{\#} \in L$.

Since we worked with locally constant domains, we also obtain the following:

Theorem

Let $L \in \{\mathbf{QH}, \mathbf{QKC}\}$. Then L is undecidable with two variables and a single monadic predicate letter.

Corollary

Let $L \in \{\mathbf{QH}, \mathbf{QKC.cd}\}$. Then L is undecidable with two variables and a single monadic predicate letter.

Open problem

Problem

What about **QLC** (*Dummett's logic*)*?*

Conjecture

QLC *is decidable with two variables.*

Similar things can be done for logics of frames with finitely many worlds, linear frames, etc. Techniques differ, but ideas are broadly similar:

- encode a suitable problem with formulas containing a few variables;
- use some form of the Kripke trick to get rid of binary letters;
- simulate all the monadic letters with a single one;
- at each stage, make sure to prepare the ground for what is to come.

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